Correction du contrôle

Du jeudi 12 octobre 2023

EXERCICE 1

QCM **(4 points)**

1) **Réponse a) :** $z = 18 + 4i \Rightarrow \bar{z} = 18 - 4i$.

2) **Réponse d) :** $\dfrac{1}{a} = \dfrac{1}{2-3i} = \dfrac{2+3i}{4+9} = \dfrac{2}{13} + \dfrac{3}{13}i$.

3) **Réponse d) :** $P(2i) = (2i)^3 - 2(2i)^2 + 2(2i) = -8i + 8 + 4i = 8 - 4i$.

4) **Réponse c) :** $z = \dfrac{4+2i}{-4+3i} = \dfrac{(4+2i)(-4-3i)}{16+9} = \dfrac{-16-12i-8i+6}{25} = -\dfrac{2}{5} - \dfrac{4}{5}i$.

EXERCICE 2

Forme algébrique **(2 points)**

1) $z_1 = \dfrac{4+3i}{1+2i} = \dfrac{(4+3i)(1-2i)}{1+4} = \dfrac{4-8i+3i+6}{5} = 2 - i$.

2) $z_2 = \dfrac{(2-5i)(1-3i)}{-1+i} = \dfrac{2-6i-5i-15}{-1+i} = \dfrac{(-13-11i)(-1-i)}{1+1} = \dfrac{13+13i+11i-11}{2} = 1 + 12i$.

EXERCICE 3

Équations du premier degré **(3 points)**

1) $(2+4i)(z-2i) = 1 \Leftrightarrow z = \dfrac{1}{2+4i} + 2i = \dfrac{1+4i-8}{2+4i} = \dfrac{(-7+4i)(2-4i)}{4+16}$

$= \dfrac{-14+28i+8i+16}{20} = \dfrac{1}{10} + \dfrac{9}{5}i \Leftrightarrow S = \left\{\dfrac{1}{10} + \dfrac{9}{5}i\right\}$.

2) $(3+2i)\,z + (1-5i)\,\bar{z} = -19-2i \Leftrightarrow (3+2i)(x+iy) + (1-5i)(x-iy) = -19-2i \Leftrightarrow$

$3x + 3iy + 2ix - 2y + x - iy - 5ix - 5y = -19-2i \Leftrightarrow 4x - 7y + i(-3x+2y) = -19-2i$

$\Leftrightarrow \begin{cases} 4x - 7y = -19 & (\times 3) \\ -3x + 2y = -2 & (\times 4) \end{cases} \Leftrightarrow \begin{cases} 12x - 21y = -57 \\ -12x + 8y = -8 \end{cases}$ par somme $-13y = -65 \Leftrightarrow y = 5$

En remplaçant dans la 1re équation : $x = \dfrac{-19+7y}{4} = \dfrac{-19+35}{4} = 4$ d'où $S = \{4+5i\}$.

EXERCICE 4

Équations du second degré **(3 points)**

1) $2z^2+11z+16 = 0$ on a $\Delta = 121-128 = -7 < 0$ deux solutions complexes conjuguées.

$$z_1 = \frac{-11+i\sqrt{7}}{4} \quad \text{et} \quad z_2 = \frac{-11-i\sqrt{7}}{4}$$

2) $(x+iy)^2 - 4(x-iy) - 5 = 0 \Leftrightarrow (x^2 - y^2 - 4x - 5) + i(2xy + 4y) = 0 \Leftrightarrow$

$$\begin{cases}(x-2)^2 - y^2 = 9 \ \ (1)\\ y(x+2) = 0 \ \ (2)\end{cases} \Leftrightarrow \begin{cases}(2)\ \ y = 0\\ (1)\ \ (x-2)^2 = 9\end{cases} \text{ ou } \begin{cases}(2)\ \ x = -2\\ (1)\ \ y^2 = 7\end{cases} \Leftrightarrow$$

$$\begin{cases}y = 0\\ x = 5 \text{ ou } x = -1\end{cases} \text{ ou } \begin{cases}x = -2\\ y = \pm\sqrt{7}\end{cases} \Leftrightarrow S = \{5\,;\,-1\,;\,-2+i\sqrt{7}\,;\,-2-i\sqrt{7}\}$$

Exercice 5

Ensemble de points **(5 points)**

1) $z' \in \mathbb{R} \Leftrightarrow z' = \overline{z'} \Leftrightarrow \dfrac{z+2i}{z-1} = \dfrac{\bar{z}-2i}{\bar{z}-1} \overset{z\neq 1}{\Leftrightarrow} (z+2i)(\bar{z}-1) = (z-1)(\bar{z}-2i) \Leftrightarrow$

$\cancel{z\bar{z}} - z + 2i\bar{z} - 2i = \cancel{z\bar{z}} - 2iz - \bar{z} + 2i \Leftrightarrow -(z-\bar{z}) + 2i(z+\bar{z}) = 4i$ on pose $z = x + iy$ avec $x, y \in \mathbb{R}$

On obtient alors : $-2iy + 4ix = 4i \overset{\div 2i}{\Leftrightarrow} -y + 2x = 2 \Leftrightarrow y = 2x - 2.$

E_1 est la droite d'équation $y = 2x - 2$ privé du point A.

2) $z' \in i\mathbb{R} \Leftrightarrow z' = -\overline{z'} \Leftrightarrow \dfrac{z+2i}{z-1} = \dfrac{-\bar{z}+2i}{\bar{z}-1} \Leftrightarrow (z+2i)(\bar{z}-1) = (z-1)(-\bar{z}+2i) \Leftrightarrow$

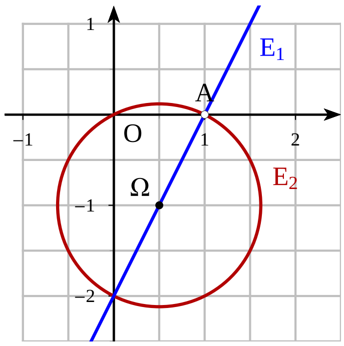
$z\bar{z} - z + 2i\bar{z} - \cancel{2i} = -z\bar{z} + 2iz + \bar{z} - \cancel{2i} \Leftrightarrow 2z\bar{z} - (z+\bar{z}) - 2i(z-\bar{z}) = 0$ on pose $z = x + iy$

On obtient alors : $2(x^2+y^2) - 2x + 4y = 0 \overset{\div 2}{\Leftrightarrow} x^2 - x + y^2 + 2y = 0 \Leftrightarrow$

$$\left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + (y+1)^2 - 1 = 0 \Leftrightarrow \left(x - \frac{1}{2}\right)^2 + (y+1)^2 = \frac{5}{4}$$

E_2 est le cercle de centre $\Omega\left(\dfrac{1}{2}\,;-1\right)$ et de rayon $\sqrt{\dfrac{5}{4}} = \dfrac{\sqrt{5}}{2}$ privé du point A.

3) On obtient les ensembles Γ_1 et Γ_2 suivants :



Exercice 6

Racines d'un polynôme **(3 points)**

1) $(z^2 + bz + 4)(z^2 - bz + 4) = z^4 + (8 - b^2)z^2 + 16 \overset{b>0}{\Rightarrow} b = 2\sqrt{2}$

2) $P(z) = 0 \Leftrightarrow (z^2 + 2z\sqrt{2} + 4)(z^2 - 2z\sqrt{2} + 4) = 0$

$z^2 + 2z\sqrt{2} + 4 = 0$
$\Delta = 8 - 16 = -8 < 0$

2 solutions complexes conjuguées

$$z_1 = \frac{-2\sqrt{2} + 2i\sqrt{2}}{2} = -\sqrt{2} + i\sqrt{2}$$
$$z_2 = \frac{-2\sqrt{2} - 2i\sqrt{2}}{2} = -\sqrt{2} - i\sqrt{2}$$

ou

$z^2 - 2z\sqrt{2} + 4 = 0$
$\Delta = 8 - 16 = -8 < 0$

2 solutions complexes conjuguées

$$z_3 = \frac{2\sqrt{2} + 2i\sqrt{2}}{2} = \sqrt{2} + i\sqrt{2}$$
$$z_4 = \frac{2\sqrt{2} - 2i\sqrt{2}}{2} = \sqrt{2} - i\sqrt{2}$$