Answers to exercises

Chapter 1

EXERCISE 1

a) $u_1 = 3, u_2 = 5, u_3 = 14, \quad u_{n+2} = 9u_n + 4$

b) We can write the following algorithm :

```
Variables: N, I  integers    U  real number
Inputs and initialization
  | Read N
  | 1 → U
Processing
  | for I from 1 to N do
  |   | 3U − 1 → U
  | end
Output  : Print U
```

n	5	10	15
u_n	122	29 252	7 174 454

c) We may propose two algorithms, one does not store all the terms of (u_n) and the other stores them in a list.

```
Variables: I : integers    U : real number
Inputs and initialization
  | 1 → U
Processing and outputs
  | for I from 1 to 10 do
  |   | 3U − 1 → U
  |   | Print U
  | end
```

```
Variables: I : integer    U : real number
           L1 : list
Inputs and initialization
  | 1 → U
  | Clear L1
Processing
  | for I from 1 to 10 do
  |   | 3U − 1 → U
  |   | U → L1(I)
  | end
Output  : Print L1
```

EXERCISE 2

a) $u_2 = 14, u_3 = 52, u_4 = 194$

b) The algorithm opposite is an example :

```
Variables: N, I  integers
           U, V, W  real numbers
Inputs and initialization
  | Read N
  | 1 → V   (u0)
  | 1 → U   (u1)
Processing
  | for I from 2 to N do
  |   | 4U − V → W
  |   | U → V
  |   | W → U
  | end
Output  : Print U
```

n	6	10
u_n	2 702	524 174

Exercise 3

a) $u_{n+1} - u_n = -3 < 0 \quad (u_n)$ is decreasing

b) $u_{n+1} - u_n = \dfrac{1}{(n+3)(n+2)} > 0 \quad (u_n)$ is decreasing

c) $\dfrac{u_{n+1}}{u_n} = 2 > 1 \quad (u_n)$ is increasing

d) (u_n) is neither increasing nor decreasing.

Exercise 4

$\dfrac{u_{n+1}}{u_n} = \dfrac{n+1}{n^2} \leqslant 1 \quad (u_n)$ is decreasing

because $n \geqslant 2 \quad \Leftrightarrow \quad n^2 \geqslant 2n \geqslant n+1$

Exercise 5

a) $\dfrac{u_{n+1}}{u_n} = \dfrac{(n+1)^2}{n^2} \leqslant 1 \quad (u_n)$ is decreasing.

because $n \geqslant 4 \quad \Leftrightarrow \quad n^2 \geqslant 4n \quad \Leftrightarrow \quad n^2 + n^2 \geqslant n^2 + 4n \quad \Leftrightarrow \quad 2n^2 \geqslant n^2 + 2n + 1$

b) $u_{n+1} - u_n = \dfrac{1}{2^{n+1}} \geqslant 0 \quad (u_n)$ is increasing

Exercise 6

$u_{n+1} - u_n = \dfrac{1}{2^{n+1}} - 1 \leqslant 0 \quad (u_n)$ is decreasing

because $2^{n+1} \geqslant 2 \quad \Leftrightarrow \quad \dfrac{1}{2^{n+1}} - 1 \leqslant \dfrac{1}{2} - 1 \leqslant -\dfrac{1}{2}$

Exercise 7

a) **True proposition :** we can add two equalities

$u_{n+1} \geqslant u_n$ and $v_{n+1} \geqslant v_n \Rightarrow u_{n+1} + v_{n+1} \geqslant u_n + v_n$

b) **False proposition :** we can multiply two inequalities only if the terms are positive. The question does not specify that this is the case.

Counterexample : Let (u_n) and (v_n) be sequences defined on $\mathbb{N}^*$ by $u_n = n^2$ and $v_n = -\dfrac{1}{n}$. Both sequences are increasing. The sequence (w_n) defined by $w_n = u_n \times v_n = -n$ is clearly decreasing.

Exercise 8

a) $u_n = 2 + \dfrac{1}{2}n$

b) $u_5 = u_2 + 3r \quad \Leftrightarrow \quad r = \dfrac{u_5 - u_2}{3} = -18$. Thus : $u_{20} = u_2 + 18r = -283$

c) $u_{20} = u_1 + 19r = 55$ then $S = 20 \times \dfrac{-2+55}{2} = 530$

d) $u_{25} = u_0 + 25r = -53$ and $u_{125} = u_0 + 125r = -253$. Thus :

$$S = 101 \times \frac{-53 - 253}{2} = -15\,453$$

Exercise 9

a) $u_1 = \frac{1}{2},\ u_2 = \frac{1}{3},\ u_3 = \frac{1}{4},\ u_4 = \frac{1}{5}.$

We can then surmise that : $u_n = \dfrac{1}{n+1}$

b) $v_{n+1} - v_n = \dfrac{1}{u_{n+1}} - \dfrac{1}{u_n} = \dfrac{1 + u_n}{u_n} - \dfrac{1}{u_n} = 1$

$\forall n \in \mathbb{N},\ v_{n+1} - v_n = 1$ the sequence (v_n) is arithmetic with a common difference of 1 and a 1st term of $v_0 = 1$

c) $v_n = 1 + n$ then $u_n = \dfrac{1}{v_n} = \dfrac{1}{n+1}$ the conjecture is verified.

Exercise 10

a) $u_n = 5\left(\dfrac{2}{5}\right)^{n-1}$

b) $u_9 = u_4 \times q^5 \ \Leftrightarrow\ q^5 = \dfrac{u_9}{u_4} = 25\sqrt{5} = \sqrt{5}^5$

then $q = \sqrt{5}$ and $u_{14} = u_4 \times q^{10} = 5^5 = 3\,125$

c) $S = u_0 \dfrac{1 - q^{13}}{1 - q}$ then $u_0 = \dfrac{S(1-q)}{1 - q^{13}} = \dfrac{24\,573}{8\,191} = 3$

Exercise 11

$\forall n \in \mathbb{N},\ \dfrac{u_{n+1}}{u_n} = \dfrac{2^{n+1}}{3^{n+2}} \times \dfrac{3^{n+1}}{2^n} = \dfrac{2}{3}$

The sequence (u_n) is geometric with a common ration of $q = \dfrac{2}{3}$ and a first term of $u_0 = \dfrac{1}{3}$

The sequence (u_n) converges to 0 because $-1 < q < 1$

Exercise 12

a) See the course

b) B is the sum of the terms of an arithmetic sequence with a common difference of $\dfrac{1}{2}$ and a first term of $\dfrac{1}{2}$

There are : $\dfrac{\frac{1}{2} + 10}{\frac{1}{2}} + 1 = 20$ terms. B = $20 \times \dfrac{\frac{1}{2} + 10}{2} = 105$

c) See the course

Exercise 13

a) $v_{n+1} = u_{n+1} - 6 = \dfrac{1}{3}u_n + 4 - 6 = \dfrac{1}{3}(u_n - 6) = \dfrac{1}{3}v_n$

$\forall n \in \mathbb{N}$, $\dfrac{v_{n+1}}{v_n} = \dfrac{1}{3}$, the sequence (v_n) is geometric with a common ratio $q = \dfrac{1}{3}$ and of 1st term $v_0 = -5$

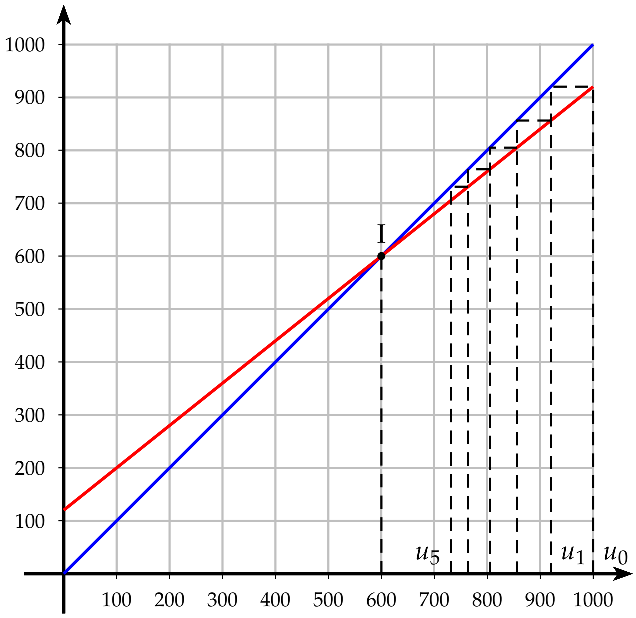
b) $v_n = -5\left(\dfrac{1}{3}\right)^n$ then $u_n = -5\left(\dfrac{1}{3}\right)^n + 6.$

c) $\lim\limits_{n\to+\infty} \left(\dfrac{1}{3}\right)^n = 0$ car $-1 < \dfrac{1}{3} < 1.$ The sequence (u_n) converges to 6.

Exercise 14

1) a) $p_{n+1} = 0,8\, p_n + 120$

b) Selecting a suitable viewing window, we plot the lines of equations $y = x$ and $y = 0,8x + 120$



The sequence (p_n) seems to be decreasing and to converge to 600.

2) a) $v_{n+1} = p_{n+1} - 600 = 0,8p_n + 120 - 600 = 0,8(p_n - 600) = 0,8v_n$

$\forall n \in \mathbb{N}$, $\dfrac{v_{n+1}}{v_n} = 0,8$, the sequence (v_n) is geometric with a common ratio of 0,8 and a 1st term of $v_0 = 400$

b) $v_n = 400 \times 0,8^n$ then $p_n = 400 \times 0,8^n + 600$

c) $\lim\limits_{n\to+\infty} 0,8^n = 0$ car $-1 < 0,8 < 1.$ The sequence(p_n) converges to 600.

Exercise 15

a) $v_{n+1} = \dfrac{u_{n+1} - 1}{u_{n+1} + 3} = \dfrac{\dfrac{2u_n + 3}{u_n + 4} - 1}{\dfrac{2u_n + 3}{u_n + 4} + 3} = \dfrac{2u_n + 3 - u_n - 4}{2u_n + 3 + 3u_n + 12} = \dfrac{u_n - 1}{5u_n + 15}$

$$v_{n+1} = \frac{u_n - 1}{5(u_n + 3)} = \frac{1}{5} v_n$$

$\forall n \in \mathbb{N},\ \dfrac{v_{n+1}}{v_n} = \dfrac{1}{5}$, the sequence (v_n) is geometric with a common ratio of $\dfrac{1}{5}$ and a 1st term of $v_0 = -\dfrac{1}{3}$

b) $v_n = -\dfrac{1}{3}\left(\dfrac{1}{5}\right)^n$

$$v_n = \frac{u_n - 1}{u_n + 3} \Leftrightarrow u_n = \frac{3v_n + 1}{1 - v_n} \Leftrightarrow u_n = \frac{-\left(\frac{1}{5}\right)^n + 1}{1 + \frac{1}{3}\left(\frac{1}{5}\right)^n} = \frac{-1 + 5^n}{5^n + \frac{1}{3}}$$

c) $\displaystyle\lim_{n \to +\infty} \left(\frac{1}{5}\right)^n = 0$ because $-1 < \dfrac{1}{5} < 1$

We then have : $\displaystyle\lim_{n \to +\infty} v_n = 0$ and $\displaystyle\lim_{n \to +\infty} u_n = 1$

Exercise 16

1) $u_2 = u_1 - \dfrac{1}{4} u_0 = \dfrac{1}{2} + \dfrac{1}{4} = \dfrac{3}{4}$

$u_2 - u_1 = \dfrac{1}{4}$ and $u_1 - u_0 = \dfrac{3}{2}$

$u_2 - u_1 \neq u_1 - u_0$, the sequence (u_n) is not arithmetic

$\dfrac{u_2}{u_1} = \dfrac{3}{2}$ and $\dfrac{u_1}{u_0} = -\dfrac{1}{2}$ then $\dfrac{u_2}{u_1} \neq \dfrac{u_1}{u_0}$,

hence the sequence (u_n) is not geometric.

2) a) $v_0 = u_1 - \dfrac{1}{2} u_0 = 1$

b) $v_{n+1} = u_{n+2} - \dfrac{1}{2} u_{n+1} = u_{n+1} - \dfrac{1}{4} u_n - \dfrac{1}{2} u_{n+1} = \dfrac{1}{2} u_{n+1} - \dfrac{1}{4} u_n$

$$v_{n+1} = \frac{1}{2}\left(u_{n+1} - \frac{1}{2} u_n\right) = \frac{1}{2} v_n$$

c) $\forall n \in \mathbb{N},\ \dfrac{v_{n+1}}{v_n} = \dfrac{1}{2}$. The sequence (v_n) is geometric with a common ratio of $\dfrac{1}{2}$ and a 1st term of 1.

d) $v_n = \left(\dfrac{1}{2}\right)^n$

3) a) $w_0 = \dfrac{u_0}{v_0} = -1$

b) $w_{n+1} = \dfrac{u_{n+1}}{v_{n+1}} = \dfrac{v_n + \frac{1}{2} u_n}{\frac{1}{2} v_n} = \dfrac{2v_n + u_n}{v_n}$

c) $w_{n+1} = \dfrac{2v_n}{v_n} + \dfrac{u_n}{v_n} = w_n + 2$

d) $\forall n \in \mathbb{N},\ w_{n+1} - w_n = 2$. The sequence (w_n) is arithmetic with a common difference of 2 and a 1st term of $w_0 = -1$. We then have : $w_n = -1 + 2n$

4) $u_n = w_n \times v_n = \dfrac{2n-1}{2^n}$

5) We can write the following algorithm :

Variables: I, N integers
Inputs and initialization
  Read N
  $-1 \to S$
Processing
  for I from 1 to N **do**
    $S + \dfrac{2I-1}{2^I} \to S$
  end
Output : Print S

n	6	10	15
u_n	1,765 6	1,977 5	1,999 0

It can be conjectured that the sequence (u_n) converges to 2

⚠ This conjecture can be proven by mathematics induction :

$$\forall n \in \mathbb{N},\ S_n = 2 - \frac{2n-3}{2^n}$$

Exercise 17

1) $10w_{10} = 11w_9 + 1 = 11 \times 19 + 1 = 210$. So $w_{10} = 21$

2) The sequence (w_n) seems to be arithmetic with a common difference of 2 and a 1st term of 1.
$w_n = 1 + 2n$ so $w_{2009} = 1 + 2 \times 2009 = 4019$

⚠ We could prove this conjecture by induction.

Exercise 18

1) Let P be a cubic polynomial, we then have :
$P(x) = ax^3 + bx^2 + cx + d$
$a \neq 0$, b, c and d are real numbers.

Calculate the quantity : $D(x) = P(x+1) - P(x)$

$$\begin{aligned} D(x) &= a(x+1)^3 + b(x+1)^2 + c(x+1) + d - (ax^3 + bx^2 + cx + d) \\ &= a(x^3 + 3x^2 + 3x + 1) + b(x^2 + 2x + 1) + cx + c + d - ax^3 - bx^2 - cx - d \\ &= ax^3 + 3ax^2 + 3ax + a + bx^2 + 2bx + b + c - ax^3 - bx^2 \\ &= 3ax^2 + (3a + 2b)x + a + b + c \end{aligned}$$

Now we want $D(x) = x^2$, by identifying the coefficients the following system is obtained :

$$\begin{cases} 3a = 1 \\ 3a + 2b = 0 \\ a + b + c = 0 \end{cases} \Leftrightarrow \begin{cases} a = \dfrac{1}{3} \\ b = -\dfrac{1}{2} \\ c = -\dfrac{1}{3} + \dfrac{1}{2} = \dfrac{1}{6} \end{cases}$$

We can therefore propose the following polynomial (taking $d = 0$) :

$$P(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}x$$

2) The equalities can be completed using the property P :

$$\begin{aligned} P(1) - P(0) &= 0^2 = 0 \\ P(2) - P(1) &= 1^2 = 1 \\ P(3) - P(2) &= 2^2 = 4 \\ \cdots &= \cdots \\ P(n+1) - P(n) &= n^2 \end{aligned}$$

3) Adding the equalities term by term :
$[P(1) - P(0)] + [P(2) - P(1)] + \cdots + [P(n+1) - P(n)] = 0^2 + 1^2 + \cdots + n^2$
All the terms cancel out except $P(n+1)$ and $P(0)$

$$P(n+1) - P(0) = 1^2 + 2^2 + \cdots + n^2$$

By calculating $P(n+1) - P(0)$ we find that

$$\begin{aligned} P(n+1) - P(0) &= \frac{1}{3}(n+1)^3 - \frac{1}{2}(n+1)^2 + \frac{1}{6}(n+1) \\ &= \frac{(n+1)[2(n+1)^2 - 3(n+1) + 1]}{6} \\ &= \frac{(n+1)(2n^2 + 4n + 2 - 3n - 3 + 1)}{6} \\ &= \frac{(n+1)(2n^2 + n)}{6} \\ &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

This completes the proof without using induction !

EXERCISE 19

a) Algorithm for calculating the slope and the constant coefficient of a line through two points A and B
(AB) : $y = mx + p$

Variables: a, b, c, d integers
m, p real numbers
Inputs and initialization
| Read a, b, c, d
Processing
$\frac{d-b}{c-a} \to m$
$b - ma \to p$
Output : Print m, p

b) We enter the program.

c) We can suggest :

Variables: a, b, c, d integers
m, p real numbers
Inputs and initialization
Read a, b, c, d
Processing and outputs
if $a = c$ **then**
Print "m does not exist"
else
$\dfrac{d-b}{c-a} \to m$
$b - ma \to p$
Print m, p
end

EXERCISE 20

No answer

EXERCISE 21

1) We have the following table :
Init : initialization, B : loop

	Init	B nº 1	B nº 2	B nº 3
u	1	3	6	11
S	1	4	10	21
i	0	1	2	3

2) We complete the table

n	0	1	2	3	4	5
u	1	3	6	11	20	37
S	1	4	10	21	41	78

3) We complete the table :

n	0	1	2	3	4	5
u_n	1	3	6	11	20	37
$u_n - n$	1	2	4	8	16	32

We can conjecture that : : $\forall n \in \mathbb{N}, \;\; u_n - n = 2^n$

4) Let prove us this conjecture. Let : $v_n = u_n - n$

$$v_{n+1} = u_{n+1} - (n+1) = 2u_n + 1 - n - n - 1 = 2u_n - 2n = 2(u_n - n) = 2v_n$$

$$\forall n \in \mathbb{N}, \;\; \frac{v_{n+1}}{v_n} = 2$$

The sequence (v_n) is geometric with a common ration of 2 and a 1st term of 1.
Then $v_n = 2^n$. The conjecture is proven.

Therefore $u_n - n = 2^n \;\Leftrightarrow\; u_n = 2^n + n$

$$\begin{aligned} S_n &= u_0 + u_1 + u_2 + \cdots + u_n \\ &= 2^0 + (2^1 + 1) + (2^2 + 2) + \cdots + (2^n + n) \\ &= (2^0 + 2^1 + 2^2 + \cdots + 2^n) + (1 + 2 + \cdots + n) \\ &= \frac{1 - 2^{n+1}}{1 - 2} + \frac{n(n+1)}{2} \\ &= 2^{n+1} - 1 + \frac{n(n+1)}{2} \end{aligned}$$

Note that (S_n) is the sum of the terms of a geometric sequence with a common ratio of 2 and of an arithmetic sequence with a common difference of 1.