Mathematical induction. Limits of sequences

Proofreading of English by Laurence Weinstock

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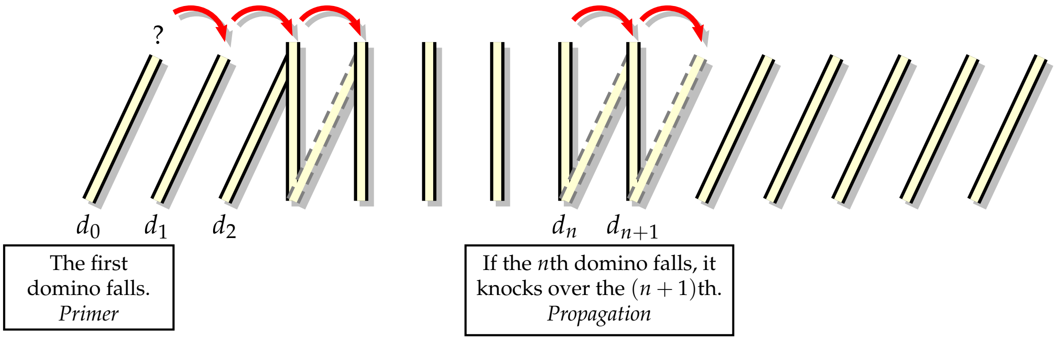
1 Mathematical induction

1.1 Domino effect or chain reaction

Mathematical induction is similar to the domino effect. It is analogous to a row of dominoes spaced regularly :



- The first domino d_0 falls. This primes the reaction.
- The dominoes are close enough together so that if one domino d_n falls the following d_{n+1} also falls. There is therefore propagation along of the row of dominoes.

So it can be concluded that all dominoes in the row fall one after the other.

Let's transpose this domino effect into a mathematical property.

Let (u_n) be the sequence defined by : $u_0 = 0,3$ and $\forall n \in \mathbb{N},\ u_{n+1} = \dfrac{1}{2}u_n + \dfrac{1}{2}$

Consider the property (P) : $\forall n \in \mathbb{N},\ 0 < u_n < 1$

- The first domino falls :
  $u_0 = 0,3$ then $0 < u_0 < 1$. The property is primed.
- If a domino falls, the following also falls :
  If $0 < u_n < 1 \Rightarrow 0 < \dfrac{1}{2}u_n < \dfrac{1}{2} \Rightarrow \dfrac{1}{2} < \dfrac{1}{2}u_n + \dfrac{1}{2} < 1$.
  We then have $0 < \dfrac{1}{2} < u_{n+1} < 1$. The property is hereditary.

As the first domino has fallen and the others fall by propagation, all the dominoes fall and therefore the property is satisfied for any natural integer.

1.2 Significance of mathematical induction

Consider the sequence (u_n) defined by : $u_0 = 0$ and $\forall n \in \mathbb{N},\ u_{n+1} = 2u_n + 1$

We would like to have a formula to explicitly compute u_n as a function of n. At first, there is no obvious formula to be seen.
In such a situation, calculating the first terms of the sequence is interesting as it often allows you to see a pattern emerge.
Let's calculate the first few terms :

$$\begin{array}{ll} u_1 = 2u_0 + 1 = 1 & (2^1 - 1) \\ u_2 = 2u_1 + 1 = 3 & (2^2 - 1) \\ u_3 = 2u_2 + 1 = 7 & (2^3 - 1) \\ u_4 = 2u_3 + 1 = 15 & (2^4 - 1) \\ u_5 = 2u_4 + 1 = 31 & (2^5 - 1) \end{array}$$

The terms of (u_n) seem to follow a simple law : by adding 1 to each term, the successive power of 2 is obtained.

From this observation, the following conjecture can be put forward : $\forall n \in \mathbb{N}, \ \ u_n = 2^n - 1$

⚠ A conjecture is not a proof (i.e. not necessarily a true affirmation, some conjectures sometimes prove to be false...). It is only the statement of a property resulting from a number of observations.

How can this conjecture be confirmed ?
Note (P) the property, defined by : $\forall n \in \mathbb{N}, \ \ u_n = 2^n - 1$
Let's assume for any arbitrary n that the property (P) is true : $u_n = 2^n - 1$

Then we have : $u_{n+1} = 2u_n + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 1$
This is the property (P) for the index $n + 1$.

If the property is true for an arbitrary index n (induction hypothesis) then the property is also true for the following index $n + 1$. This is called the **induction step**.

We have checked that the property is true for the indices 0, 1, 2, 3, 4, 5. This is called **the basis step**. But with the induction step, it will also be true for the index $n = 6$, then for the index $n = 7$ etc. The property is therefore true for all n.

1.3 Axiom of induction

Definition 1 : Consider the property (P_n) defined on $\mathbb{N}$.

- If the property is satisfied for the first index 0 or n_0 : *basis step*
- and if the induction step is true from index 0 or n_0, i.e. :
  $\forall n \geqslant 0$ or $n \geqslant n_0$ then $\mathrm{P}_n \Rightarrow \mathrm{P}_{n+1}$

then the property is true for all n from index 0 or n_0

Note : Mathematical induction is like the domino effect :

If a domino falls, then the next one falls.

If the first falls then the second falls, then the third, then the fourth...

Conclusion : if the first domino falls, then they all fall.

Mathematical induction has two steps :
- Prove the basis step
- Prove the induction step

If we can carry out these two steps then the property is proven for all natural numbers.

⚠ It must be ensured that the two conditions "basis step" and "induction step" are proven. As a matter of fact, if one of the two conditions is not met, then an erroneous conclusion is made, as shown by two examples in paragraph 1.6.

1.4 Bernoulli's inequality

Theorem 1 : Let a be a strictly positive real number. We then have

$$\forall n \in \mathbb{N}, \quad (1+a)^n \geqslant 1 + na$$

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Prove this inequality by mathematical induction.

- **Basis step :**
  $(1+a)^0 = 1$ and $1 + 0a = 1$, then $(1+a)^0 \geqslant 1 + 0 \times a$. The basis step is established.
- **Induction step :**
  Supposing that $(1+a)^n \geqslant 1 + na$, let's show that $(1+a)^{n+1} \geqslant 1 + (n+1)a$

  By hypothesis : $(1+a)^n \geqslant 1 + na$ as $1 + a > 0$ and as a result :

$$\begin{aligned}(1+a)(1+a)^n &\geqslant (1+a)(1+na)\\(1+a)^{n+1} &\geqslant 1 + na + a + na^2\\&\geqslant 1 + (n+1)a + na^2 \geqslant 1 + (n+1)a\end{aligned}$$

  The induction step is established.

By reason of both the basis and induction steps : $\forall n \in \mathbb{N}, \quad (1+a)^n \geqslant 1 + na$

Note : For the induction step, the inequality is proven by "transitivity" :

$$a > b \text{ and } b > c \text{ so } a > c$$

1.5 Application to sequences

Let (u_n) be a sequence defined by : $u_0 = 1$ and $\forall n \in \mathbb{N}, \; u_{n+1} = \sqrt{2 + u_n}$

a) Prove for all n, $0 < u_n < 2$

b) Prove that the sequence is strictly increasing.

a) Let's prove by mathematical induction that u_n is bounded.

Basis step : We know that $u_0 = 1$ so $0 < u_0 < 2$. The basis step is established.

Induction step : Supposing that $0 < u_n < 2$, show that $0 < u_{n+1} < 2$.

$$0 < u_n < 2 \quad \Rightarrow \quad 2 < u_n + 2 < 4$$

As the square root function is increasing on $\mathbb{R}_+$,

$$\sqrt{2} < \sqrt{u_n + 2} < 2 \quad \Rightarrow \quad 0 < \sqrt{2} < u_{n+1} < 2$$

The induction step is established.

Therefore, by reason of both the basis and induction steps, we can conclude that :
$\forall n \in \mathbb{N}, \; 0 < u_n < 2$.

b) Let's show by mathematical induction that the sequence (u_n) is increasing.

Basis step : we know that $u_1 = \sqrt{3}$ so $u_1 > u_0$. The basis step is established.

Induction step : Supposing $u_{n+1} > u_n$, show that $u_{n+2} > u_{n+1}$.

$$u_{n+1} > u_n \quad \Rightarrow \quad u_{n+1} + 2 < u_n + 2$$

As the square root function is increasing on $\mathbb{R}_+$,

$$\sqrt{u_{n+1} + 2} > \sqrt{u_n + 2} \quad \Rightarrow \quad u_{n+2} < u_{n+1}$$

The induction step is established.

By reason of both the basis and induction steps, the sequence (u_n) is increasing.

1.6 Situations leading to an erroneous conclusion

- **Situation 1 :** Only the induction step established.

  Consider the property : $\forall n \in \mathbb{N},\ 3$ divides 2^n
  Induction step : Supposing that 3 divides 2^n, show that 3 divides 2^{n+1}.

  If 3 divides 2^n, then there exists a natural number k such that : $2^n = 3k$
  By multiplying by 2 : $2^{n+1} = 2 \times 3k = 3(2k)$. So therefore 3 divides 2^{n+1}

  Conclusion : The induction step is established but not the basis step, so it cannot be concluded that the property is true. That is fortunate, because this property is false !

- **Situation 2 :** The basis step is established up to a certain index.

  Consider the following property : $\forall n \in \mathbb{N},\ n^2 - n + 41$ is a prime number.

  The basis step is established because for $n = 0$ we obtain 41 which is a prime number.

  But there is no mathematical induction even though $\mathcal{P}(n)$ is true up to $n = 40$. This can be seen with a table of prime numbers and a list of the first terms of the sequence (u_n) defined by $u_n = n^2 - n + 41$.

n	u_n	n	u_n	n	u_n	n	u_n
0	41	11	151	22	503	33	1097
1	41	12	173	23	547	34	1163
2	43	13	197	24	593	35	1231
3	47	14	223	25	641	36	1301
4	53	15	251	26	691	37	1373
5	61	16	281	27	743	38	1447
6	71	17	313	28	797	39	1523
7	83	18	347	29	853	40	1601
8	97	19	383	30	911		
9	113	20	421	31	971		
10	131	21	461	32	1033		

  For $n = 41$, we have : $41^2 - 41 + 41 = 41^2$ which is not a prime number. The property is false.

  Conclusion : The truth of a proposition for the first set of values does not make a general rule !

2 Limit of a sequence

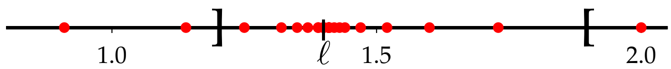
2.1 Finite limit

Definition 2 : The sequence (u_n) is said to have a limit ℓ if, and only if, all open intervals containing ℓ contain all the terms of the sequence from a certain index onwards.

The limit is denoted : $\lim\limits_{n \to +\infty} u_n = \ell$ and the sequence (u_n) is said to **converge** to ℓ

Note : If the limit exists, it is unique (This is easily proven by contradiction).

This definition conveys the idea of the accumulation of terms u_n around ℓ



Consequently The sequences defined for all natural numbers $n \neq 0$ by :

$$u_n = \frac{1}{n}, \quad v_n = \frac{1}{n^2}, \quad w_n = \frac{1}{n^3}, \quad t_n = \frac{1}{\sqrt{n}}, \quad \text{have a limit of } 0$$

Algorithm : Determine the integer N from which u_n is in an interval containing ℓ.

Consider (u_n) defined by :

$$\begin{cases} u_0 = 0,1 \\ u_{n+1} = 2u_n(1 - u_n) \end{cases}$$

This sequence converges to $\ell = 0,5$. We want to find the integer N from which the terms of the sequence are in the open interval centered at 0,5 and with a radius of 10^{-3}.

The following program will display N, with a "while" loop. We then obtain :

$N = 5$ and $|u_5 - 0,5| = 3,96\,10^{-4}$

Variables: N : integer U : real number
Inputs and initialization
 $0,1 \to U$
 $0 \to N$
Processing
 while $|U - 0,5| \geqslant 10^{-3}$ **do**
  $2U(1 - U) \to U$
  $N + 1 \to N$
 end
Output : Print N, $|U - 0,5|$

2.2 Infinite limit

Definition 3 : The sequence (u_n) is said to have a limit $+\infty$ (resp. $-\infty$) if, and only if, every interval $]A; +\infty[$ (resp. $]-\infty; B[$) contains all terms of the sequence from a certain index.

The limit is denoted : $\lim\limits_{n \to +\infty} u_n = +\infty$ resp. $\lim\limits_{n \to +\infty} u_n = -\infty$

The sequence is said to **diverge** to $+\infty$ (resp. $-\infty$)

Note : This definition conveys the idea that the terms of the sequence will always exceed the number A, no matter how large.

A sequence can have no limit. For example : $u_n = (-2)^n$. It is said that the sequence diverges.

Consequently The sequences defined for all natural numbers by :

$$u_n = n, \quad v_n = n^2, \quad w_n = n^3, \quad t_n = \sqrt{n}, \quad \text{have a limit of } +\infty$$

Algorithm : Determine the number N from which u_n is greater than a given number A (increasing sequence).

Consider the sequence (u_n) defined by :

$$\begin{cases} u_0 = -2 \\ u_{n+1} = \dfrac{4}{3}u_n + 1 \end{cases}$$

We can show that this sequence is increasing and diverges to $+\infty$. We want find the integer N from which u_n is greater than 10^3

The following program allows you to find N, by means of a "while loop".

We find that :

$N = 25$ and $U = 1325,83$

Variables: N : integer
U : real number
Inputs and initialization
$-2 \to U$
$0 \to N$
Processing
while $U \leqslant 10^3$ **do**
$\frac{4}{3}U + 1 \to U$
$N + 1 \to N$
end
Output : Print N, U

2.3 Limit by comparison and on an interval

Theorem 2 : Let (u_n), (v_n) and (w_n) be three sequences. If, from a certain index :

1) **"Squeeze" theorem**

$$v_n \leqslant u_n \leqslant w_n \text{ and if } \lim_{n\to+\infty} v_n = \ell \text{ and } \lim_{n\to+\infty} w_n = \ell \text{ then } \lim_{n\to+\infty} u_n = \ell$$

2) **Comparison theorem**

- $u_n \geqslant v_n$ and if $\lim\limits_{n\to+\infty} v_n = +\infty$ then $\lim\limits_{n\to+\infty} u_n = +\infty$
- $u_n \leqslant w_n$ and if $\lim\limits_{n\to+\infty} w_n = -\infty$ then $\lim\limits_{n\to+\infty} u_n = -\infty$

OPK **Proof** : Only the proof of the comparison theorem is on the syllabus.

We know that : $\lim\limits_{n\to+\infty} v_n = +\infty$, so for any real number A, there is an integer N such that if $n > N$ then $v_n \in]A; +\infty[$

As $u_n > v_n$ from the index p so if $n > \max(N, p)$ then $u_n \in]A; +\infty[$

Hence we have : $\lim\limits_{n\to+\infty} u_n = +\infty$

Examples :

- Prove that the sequence (u_n) defined by : $u_n = \dfrac{\sin n}{n+1}$ converges.

$$\forall n \in \mathbb{N}, \quad -\frac{1}{n+1} \leqslant \frac{\sin n}{n+1} \leqslant \frac{1}{n+1}$$

$$\text{however} \quad \lim_{n\to+\infty} -\frac{1}{n+1} = \lim_{n\to+\infty} \frac{1}{n+1} = 0$$

So, according to the squeeze theorem : $\lim_{n \to +\infty} u_n = 0$

- Show that the sequence (v_n) defined by : $v_n = n + \sin n$ diverges to $+\infty$

$$\begin{array}{rl} \forall n \in \mathbb{N} & n + \sin n \geqslant n - 1 \\ \text{however} & \lim_{n \to +\infty} n - 1 = +\infty \end{array}$$

so according to the comparison theorem : $\lim_{n \to +\infty} v_n = +\infty$

2.4 Operations on limits

You are not required to prove the following theorems. It is fairly intuitive that the limit of the sum is the sum of the limits, and likewise that the limit of the product (or quotient) is the product (or quotient) of the limits. Only 4 cases have indeterminate forms. When faced with an indeterminate form, the best approach is to try to change the form of the sequence, or use the comparison or squeeze theorems or the theorem on monotonic sequences (see further on) in order to reach a conclusion.

2.4.1 Limit of a sum

If (u_n) has a limit	ℓ	ℓ	ℓ	$+\infty$	$-\infty$	$+\infty$
If (v_n) has a limit	ℓ'	$+\infty$	$-\infty$	$+\infty$	$-\infty$	$-\infty$
then $(u_n + v_n)$ has a limit	$\ell + \ell'$	$+\infty$	$-\infty$	$+\infty$	$-\infty$	I.F.

Note : I.F. = Indeterminate form

Examples : Determine the limits of the following sequences :

- $\forall n \in \mathbb{N}^*, \; u_n = 3n + 1 + \dfrac{2}{n}$

$$\left.\begin{array}{l} \lim_{n \to +\infty} 3n + 1 = +\infty \\ \lim_{n \to +\infty} \dfrac{2}{n} = 0 \end{array}\right\} \text{The sum is } \lim_{n \to +\infty} u_n = +\infty$$

- $\forall n \in \mathbb{N}^*, \; v_n = \left(\dfrac{1}{3}\right)^n + 5 - \dfrac{1}{n}$

$$\left.\begin{array}{l} \lim_{n \to +\infty} \left(\dfrac{1}{3}\right)^n = 0 \\ \lim_{n \to +\infty} 5 - \dfrac{1}{n} = 5 \end{array}\right\} \text{The sum is } \lim_{n \to +\infty} v_n = 5$$

- $\forall n \in \mathbb{N}, \; w_n = n^2 - n + 2$

$$\left.\begin{array}{l} \lim_{n \to +\infty} n^2 = +\infty \\ \lim_{n \to +\infty} -n + 2 = -\infty \end{array}\right\} \text{I.F. Another method must be used}$$

2.4.2 Limit of a product

If (u_n) has a limit	ℓ	$\ell \neq 0$	0	∞
If (v_n) has a limit	ℓ'	∞	∞	∞
then $(u_n \times v_n)$ has a limit	$\ell \times \ell'$	∞*	I.F.	∞*

*Follow the usual rules on multiplying unlike or like signs

Examples : Determine the limits of the following sequences :

a) $\forall n \in \mathbb{N}^*,\ u_n = n^2 - n + 2$
$= n^2\left(1 - \frac{1}{n} + \frac{2}{n^2}\right)$

$\lim\limits_{n\to+\infty} n^2 = +\infty$
$\lim\limits_{n\to+\infty} 1 - \frac{1}{n} + \frac{2}{n^2} = 1$

The product is $\lim\limits_{n\to+\infty} u_n = +\infty$

$\forall n \in \mathbb{N},\ v_n = (2-n)\times 3^n$

$\lim\limits_{n\to+\infty} 3^n = +\infty$
$\lim\limits_{n\to+\infty} 2 - n = -\infty$

The product is $\lim\limits_{n\to+\infty} v_n = -\infty$

$\forall n \in \mathbb{N},\ w_n = \frac{1}{n+1}\times(n^2+3)$

$\lim\limits_{n\to+\infty} \frac{1}{n+1} = 0$
$\lim\limits_{n\to+\infty} n^2 + 3 = +\infty$

I.F. Expressed in this form

2.4.3 Limit of a quotient

If (u_n) has a limit	ℓ	$\ell \neq 0$	0	ℓ	∞	∞
If (v_n) has a limit	$\ell' \neq 0$	0 (1)	0	∞	ℓ'	∞
then $\left(\frac{u_n}{v_n}\right)$ has a limit	$\frac{\ell}{\ell'}$	∞*	I.F.	0	∞*	I.F.

*Follow the usual rules on dividing like or unlike signs (1) 0 without changing sign

Examples : Determine the limits of the following sequences :

a) $\forall n \in \mathbb{N},\ u_n = \frac{5}{2n^2+1}$

$\lim\limits_{n\to+\infty} 5 = 5$
$\lim\limits_{n\to+\infty} 2n^2 + 3 = +\infty$

The quotient is $\lim\limits_{n\to+\infty} u_n = 0$

b) $\forall n \in \mathbb{N},\ v_n = \frac{1-n}{0,5^n}$

$\lim\limits_{n\to+\infty} 1 - n = -\infty$
$\lim\limits_{n\to+\infty} 0,5^n = 0^+$

The quotient is $\lim\limits_{n\to+\infty} v_n = -\infty$

c) $\forall n \in \mathbb{N}^*,\ w_n = \frac{n^2+3}{n+1} = \frac{n + \frac{3}{n}}{1 + \frac{1}{n}}$

Factoring out n

$\lim\limits_{n\to+\infty} n + \frac{3}{n} = +\infty$
$\lim\limits_{n\to+\infty} 1 + \frac{1}{n} = 1$

The quotient is $\lim\limits_{n\to+\infty} w_n = +\infty$

2.5 Limit of a geometric sequence

Theorem 3 : Let q be a real number. Consider the following sequences :

- If $q > 1$ then $\lim_{n \to +\infty} q^n = +\infty$
- If $q = 1$ then $\lim_{n \to +\infty} q^n = 1$
- If $-1 < q < 1$ then $\lim_{n \to +\infty} q^n = 0$
- If $q \leqslant -1$ then $\lim_{n \to +\infty} q^n$ does not exist

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Proof : Only the proof of the first limit is on the syllabus.
Bernoulli's inequality is proven by mathematical induction. Therefore for all $a > 0$

$$\forall n \in \mathbb{N}, \quad (1+a)^n \geqslant 1 + na$$

Let $q = 1 + a$ so if $a > 0$ then $q > 1$. The inequality becomes :

$$q^n \geqslant 1 + na$$

Seeing as $a > 0$ we have : $\lim_{n \to +\infty} 1 + na = +\infty$

According to the comparison theorem : $\lim_{n \to +\infty} q^n = +\infty$

Note : To prove the third limit, let $Q = \dfrac{1}{|q|}$, with $0 < |q| < 1$ therefore $Q > 1$.
By taking the limit on each side of the equality we can conclude with the quotient of the limits.

Example : Consider the sequence (u_n) defined by : $\begin{cases} u_0 = 2 \\ u_{n+1} = 2u_n + 5 \end{cases}$

Let us define the sequence (v_n) such that $v_n = u_n + 5$

1) Show that the sequence (v_n) is geometric
2) Express v_n then u_n in terms of n
3) Deduce the limit of (u_n)

⇒ ⇒

1) We have to show that $\forall x \in \mathbb{N} \quad v_{n+1} = qv_n$

$$v_{n+1} = u_{n+1} + 5 = (2u_n + 5) + 5 = 2(u_n + 5) = 2v_n$$

Therefore (v_n) is a geometric sequence with a common ratio of $q = 2$ and a 1st term of $v_0 = u_0 + 5 = 7$

2) It can therefore be deduced that : $v_n = v_0\, q^n = 7 \times 2^n$ so $u_n = v_n - 5 = 7 \times 2^n - 5$

3) According to the aforementioned theorem , $2 > 1$, so $\lim_{n \to +\infty} 2^n = +\infty$

Using the sum and product of limits, we can conclude that : $\lim_{n \to +\infty} u_n = +\infty$

2.6 Convergence of a monotonic sequence

2.6.1 Upper-bounded, lower-bounded and bounded sequences

Definition 4 : The sequence (u_n) is said to be bounded above if, and only if, there is a real number M such that :

$$\forall n \in \mathbb{N} \quad u_n \leqslant M$$

The sequence (u_n) is said to be bounded below if, and only if, there is a real number m such that :

$$\forall n \in \mathbb{N} \quad u_n \geqslant m$$

The sequence (u_n) is said to be is bounded if it has an upper bound and a lower bound.

Example : Show that the sequence (u_n) defined on $\mathbb{N}^*$ by :

$$u_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \quad \text{is bounded on the interval } \left[\frac{1}{2};1\right]$$

$$\begin{aligned}
\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} &\leqslant \overbrace{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}^{n \text{ terms}} \\
\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} &\leqslant n \times \frac{1}{n} \\
\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} &\leqslant 1
\end{aligned}$$

$$\begin{aligned}
\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} &\geqslant \overbrace{\frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}}^{n \text{ terms}} \\
\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} &\geqslant n \times \frac{1}{2n} \\
\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} &\geqslant \frac{1}{2}
\end{aligned}$$

So therefore : $\dfrac{1}{2} \leqslant u_n \leqslant 1$

2.6.2 Convergence theorems

Theorem 4 : **Divergence**

- If a sequence (u_n) is **increasing and not bounded above** then the sequence (u_n) diverges to $+\infty$.
- If a sequence (u_n) is **decreasing and not bounded below** then the sequence (u_n) diverges to $-\infty$.

OPK **Proof :** Let (u_n) be an increasing sequence with no upper bound.
(u_n) is not bounded above, so for all intervals $]A;+\infty[$,

$$\exists N \in \mathbb{N} \quad \text{such that :} \quad u_N \in]A;+\infty[$$

Seeing as (u_n) is increasing : $\forall n > N \quad \text{so} \quad u_n > u_N$

Hence : $\forall n > N \quad \text{and therefore} \quad u_n \in]A;+\infty[$

So from a certain index, all the terms of the sequence are in the interval $]A;+\infty[$.
The sequence (u_n) diverges to $+\infty$.

Example : : Consider (u_n) defined by : $u_0 = 1$ and $u_{n+1} = u_n + 2n + 3$.

It can easily be shown that the sequence (u_n) is increasing and by induction that $u_n \geqslant n^2$ (see exercise). So this sequence diverges to $+\infty$

⚠ The converse of this theorem is false, if a sequence diverges to $+\infty$,it is not necessarily increasing. To prove this, let us consider two sequences that diverge to $+\infty$ and that are not monotonic :

$$u_n = n + (-1)^n \qquad \text{and} \qquad \begin{cases} v_n = n & \text{if } n \text{ is even} \\ v_n = 2n & \text{if } n \text{ is odd} \end{cases}$$

Theorem 5 : Convergence

- If a sequence (u_n) is **increasing and bounded above** then the sequence (u_n) converges.
- If a sequence (u_n) is **decreasing and bounded below** then the sequence (u_n) converges.

Note : The proof of this theorem is not on the syllabus.

⚠ This theorem allows us to show that a sequence converges to a limit but does not give the value of this limit.
We can only say that if (u_n) is increasing and bounded above by M then $\ell \leqslant M$ and if (u_n) is decreasing and bounded below by m then $\ell \geqslant m$

Example : Consider (u_n) defined by :

$$\begin{cases} u_0 = 0 \\ u_{n+1} = \sqrt{3u_n + 4} \end{cases}$$

1) Show that the sequence (u_n) is increasing and bounded above by 4.

2) Deduce that the sequence (u_n) converges. By using an algorithm we can conjecture that (u_n) converges to 4 and determine the integer N from which $u_n > 3,99$.

1) Show by mathematical induction that the sequence (u_n) is increasing and bounded above by 4, i.e. :

$$\forall n \in \mathbb{N},\ 0 \leqslant u_n \leqslant u_{n+1} \leqslant 4$$

Basis step : $u_0 = 0$ and $u_1 = \sqrt{4} = 2$, we therefore find that : $0 \leqslant u_0 \leqslant u_1 \leqslant 4$

The basis step is established.

Induction step : : Supposing that $0 \leqslant u_n \leqslant u_{n+1} \leqslant 4$, show that $0 \leqslant u_{n+1} \leqslant u_{n+2} \leqslant 4$.

$$0 \leqslant u_n \leqslant u_{n+1} \leqslant 4$$
$$0 \leqslant 3u_n \leqslant 3u_{n+1} \leqslant 12$$
$$4 \leqslant 3u_n + 4 \leqslant 3u_{n+1} + 4 \leqslant 16$$

As the square root function is increasing on $\mathbb{R}_+$

$$2 \leqslant \sqrt{3u_n + 4} \leqslant \sqrt{3u_{n+1} + 4} \leqslant 4$$
$$0 \leqslant 2 \leqslant u_{n+1} \leqslant u_{n+2} \leqslant 4$$

The induction step is established.

By reason of the basis and induction steps, the sequence (u_n) is increasing and bounded above by 4.

2) (u_n) is increasing and bounded above by 4, according to the theorem of monotonic sequences, (u_n) is convergent.

The algorithm shown opposite relates to the exercise in paragraph 2.1.
We want to find the index N from which $u_n > 3,99$, then $|u_n - 4| < 10^{-2}$

We find that : $N = 7$ and $|u_7 - 4| \simeq 0,007$ thus $u_7 \simeq 3,993$

```
Variables: N : integer    U : real number
Inputs and initialization
    0 → U
    0 → N
Processing
    while |U − 4| ⩾ 10^(−2) do
        √(3U + 4) → U
        N + 1 → N
    end
Output   : Print N, |U − 4|
```

2.7 The method of Heron of Alexandria (1st century)

This is a method of calculating an approximation of a square root.

It depends on having a first approximate value of $\sqrt{A}$ written a.

$$\text{Si} \quad a < \sqrt{A} \quad \Rightarrow \quad \frac{1}{a} > \frac{1}{\sqrt{A}} \quad \Rightarrow \quad \frac{A}{a} > \frac{A}{\sqrt{A}} \quad \Rightarrow \quad \frac{A}{a} > \sqrt{A}$$

So : $a < \sqrt{A} < \dfrac{A}{a}$

Similarly, we can show that if $a > \sqrt{A}$ then $\dfrac{A}{a} < \sqrt{A} < a$

Algorithm : Knowing the value of a, we can determine an bounded interval containing $\sqrt{A}$. The interval is then reduced by taking the average m of the values a and $\dfrac{A}{a}$. We then have :

$$m = \frac{1}{2}\left(a + \frac{A}{a}\right)$$

The value of a is then replaced by the value of m and the process is repeated.

Let us construct a sequence (u_n) defined on $\mathbb{N}$ by : $\begin{cases} u_0 = a \\ u_{n+1} = \dfrac{1}{2}\left(u_n + \dfrac{A}{u_n}\right) \end{cases}$

The sequence converges very quickly : after each iteration, the number of correct decimals is doubled !

However, in the first century, nobody had ever heard of the decimal system or even the number zero. Calculating these fractions was certainly complicated !
The process can be simplified with the following program :

The first term is determined by searching for the nearest square A with a "while loop", that value is assigned to U, it takes N iterations to find u_N.

The approximate value of $\sqrt{431}$ with 2 iterations is :

$$A = 431, \quad N = 2$$

We then find :

$$U = \frac{1\,380\,161}{66\,480} \simeq 20{,}760\,544$$

The absolute error is : $|U - \sqrt{431}| \simeq 5\,10^{-6}$

Variables: A, N, I integers
U : real number
Inputs and initialization
Input A, N
$0 \to I$
while $A > I^2$ **do**
 $I + 1 \to I$
end
$I - 1 \to U$
Processing
for I from 1 to N **do**
 $\dfrac{1}{2}\left(U + \dfrac{A}{U}\right) \to U$
end
Output : Print U, $|U - \sqrt{A}|$