Answers to exercises

Chapter 2

EXERCISE 1

Basis step : $4^0 + 5 = 1 + 5 = 6$ divisible by 3. The basis step is established.

Induction step : Supposing that $4^n + 5$ is divisible by 3, show that $4^{n+1} + 5$ is divisible by 3.

$4^{n+1} + 5 = 4 \times 4^n + 5 = (3+1) \times 4^n + 5 = 3 \times 4^n + (4^n + 5)$

3×4^n is divisible by 3 and $4^n + 5$ is divisible by 3 with the induction hypothesis. The sum, $4^{n+1} + 5$ is divisible by 3. The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :
$\forall n \in \mathbb{N},\ 4^n + 5$ is divisible by 3.

EXERCISE 2

Basis step : $3^0 - 1 = 1 - 1 = 0$ divisible by 8. The basis step is established.

Induction step : Supposing that $3^{2n} - 1$ is divisible by 8, show that $3^{2(n+1)} - 1$ is divisible by 8.

$3^{2(n+1)} - 1 = 3^2 \times 3^{2n} - 1 = (8+1) \times 3^{2n} - 1 = 8 \times 3^{2n} + (3^{2n} - 1)$

8×3^{2n} is divisible by 8 et $3^{2n} - 1$ is divisible by 8 with the induction hypothesis. The sum, $3^{2(n+1)} - 1$ is divisible by 8. The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :
$\forall n \in \mathbb{N},\ 3^{2n} - 1$ is divisible by 8.

EXERCISE 3

Basis step : $1^3 + 2 \times 1 = 3$ multiple of 3. The basis step is established.

Induction step : Supposing that $n^3 + 2n$ is multiple of 3, show that $(n+1)^3 + 2(n+1)$ is multiple of 3.

$$\begin{aligned}(n+1)^3 + 2(n+1) &= n^3 + 3n^2 + 3n + 1 + 2n + 2\\ &= (n^3 + 2n) + 3n^2 + 3n + 3\\ &= (n^3 + 2n) + 3(n^2 + n + 1)\end{aligned}$$

$n^3 + 2n$ is a multiple of 3 with the induction hypothesis and $3(n^2 + n + 1)$ is a multiple of 3. The sum, $(n+1)^3 + 2(n+1)$ is a multiple of 3. The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :
$\forall n \geqslant 1,\ n^3 + 2n$ is a multiple of 3.

EXERCISE 4

Basis step : $3^1 + 2^2 = 3 + 4 = 7$ is a multiple of 7. The basis step is established.

Induction step : Supposing that $3^{2n+1}+2^{n+2}$ is a multiple of 7, show that $3^{2(n+1)+1}+2^{n+3}$ is a multiple of 7.

$$\begin{aligned}3^{2(n+1)+1}+2^{n+3} &= 3^2\times 3^{2n+1}+2\times 2^{n+2}\\ &= (7+2)3^{2n+1}+2\times 2^{n+2}\\ &= 7\times 3^{2n+1}+2(3^{2n+1}+2^{n+2})\end{aligned}$$

$7\times 3^{2n+1}$ is a multiple of 7 and $2(3^{2n+1}+2^{n+2})$ is a multiple of 7 with the induction hypothesis. The sum, $3^{2(n+1)+1}+2^{n+3}$ is a multiple of 7. The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :
$\forall n\in\mathbb{N},\ 3^{2n+1}+2^{n+2}$ is a multiple of 7.

Exercise 5

a) $S_1=1\,;\quad S_2=5\,;\quad S_3=14\,;\quad S_4=30$
$\forall n\geqslant 1,\ S_{n+1}=S_n+(n+1)^2$

b) **Basis step :** $\dfrac{1(1+1)(2+1)}{6}=1=S_1$. The basis step is established.

Induction step : Supposing that : $S_n=\dfrac{n(n+1)(2n+1)}{6}$

show that : $S_{n+1}=\dfrac{(n+1)(n+2)(2n+3)}{6}$

$$\begin{aligned}S_{n+1}=S_n+(n+1)^2 &= \frac{n(n+1)(2n+1)}{6}+(n+1)^2\\ &= \frac{(n+1)\left[n(2n+1)+6(n+1)\right]}{6}=\frac{(n+1)(2n^2+7n+6)}{6}\\ &= \frac{(n+1)(n+2)(2n+3)}{6}\end{aligned}$$

because $(n+2)(2n+3)=2n^2+3n+4n+6=n^2+7n+6$

The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :

$\forall n\geqslant 1,\ S_n=\dfrac{n(n+1)(2n+1)}{6}$

Exercise 6

a) $S_1=1\,;\quad S_2=9\,;\quad S_3=36\,;\quad S_4=100$
$\forall n\geqslant 1,\ S_{n+1}=S_n+(n+1)^3$

b) **Basis step :** $\dfrac{1^2(1+1)^2}{4}=1=S_1$. The basis step is established.

Induction step : Supposing that $S_n=\dfrac{n^2(n+1)^2}{4}$ show that $S_{n+1}=\dfrac{(n+1)^2(n+2)^2}{4}$

$$
\begin{aligned}
S_{n+1} = S_n + (n+1)^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= \frac{(n+1)^2\left[n^2+4(n+1)\right]}{4} = \frac{(n+1)^2(n^2+4n+4)}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4}
\end{aligned}
$$

The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :
$\forall n \geqslant 1, \; S_n = \dfrac{n^2(n+1)^2}{4}$

EXERCISE 7

Basis step : $1! = 1 \geqslant 2^0$. The basis step is established.

Induction step : Supposing that $n! \geqslant 2^{n-1}$
show that $(n+1)! \geqslant 2^n$

$n! \geqslant 2^{n-1} \quad \Rightarrow \quad (n+1) \times n! \geqslant (n+1) \times 2^{n-1} \quad \Rightarrow (n+1)! \geqslant 2^n$

because if $n \geqslant 1 \quad \Rightarrow \quad (n+1) \times 2^{n-1} \geqslant 2 \times 2^{n-1}$

The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :
$\forall n \geqslant 1, \; n! \geqslant 2^{n-1}$

EXERCISE 8

Basis step : $0 < u_0 < 1$. The basis step is established.

Induction step : Supposing that $0 < u_n < 1$ show that $0 < u_{n+1} < 1$.

Let f be the associated function defined on]0 ;2[by : $f(x) = x(2-x)$

We can sum up the behavior of the function with the following table : knowing that the zeros of the function f are 0 and 2 and that the quadratic coefficient is -1 :

x	0	1	2
$f(x)$	0 ↗	1	↘ 0

Therefore
$\forall x \in]0;1[, \; f(x) \in]0;1[$, we deduce that if $0 < u_n < 1 \Rightarrow 0 < f(u_n) < 1$
(It is said that the interval]0 ;1[is stable by f)

The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :
$\forall n \in \mathbb{N}, \; 0 < u_n < 1$

EXERCISE 9

Prove the proposal is equivalent to : $\forall n \in \mathbb{N}, \; 0 < u_n < u_{n+1} < 2$

Basis step : $u_0 = 1$ and $u_1 = \sqrt{3}$, so we have $0 < u_0 < u_1 < 2$. The basis step is established.

Induction step : Supposing that $0 < u_n < u_{n+1} < 2$
show that $0 < u_{n+1} < u_{n+2} < 2$.

$$0 < u_n < u_{n+1} < 2 \Rightarrow 2 < u_n + 2 < u_{n+1} + 2 < 4$$

as the square function is increasing on $\mathbb{R}_+$,

$$\sqrt{2} < \sqrt{u_n+2} < \sqrt{u_{n+1}+2} < 2 \Rightarrow$$
$$0 < \sqrt{2} < u_{n+1} < u_{n+2} < 2$$

The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :
$\forall n \in \mathbb{N},\ 0 < u_n < u_{n+1} < 2$

Exercise 10

Basis step : $u_0 = 1 = 2^0$ and $u_1 = 2 = 2^1$. The basis step is established.

Induction step : Supposing that $u_n = 2^n$ and $u_{n+1} = 2^{n+1}$, show that $u_{n+2} = 2^{n+2}$

$$u_{n+2} = 5u_{n+1} - 6u_n = 5 \times 2^{n+1} - 6 \times 2^n$$
$$= 10 \times 2^n - 6 \times 2^n = 4 \times 2^n = 2^{n+2}$$

The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :
$\forall n \in \mathbb{N},\ u_n = 2^n$

⚠ It is important to check the basis step with the first two terms, because in the induction hypothesis, the property is used for the two previous terms u_{n+2}

Exercise 11

a) $u_2 = \dfrac{1}{2}$; $u_3 = \dfrac{2}{3}$; $u_4 = \dfrac{3}{4}$; $u_5 = \dfrac{4}{5}$

b) We can conjecture that : $u_n = \dfrac{n-1}{n}$ which is satisfied from u_1 to u_5.

c) **Basis step :** $\dfrac{1-1}{1} = 0 = u_1$. The basis step is established.

Induction step : Supposing that $u_n = \dfrac{n-1}{n}$ Show that $u_{n+1} = \dfrac{n}{n+1}$

$$u_{n+1} = \frac{1}{2 - u_n} = \frac{1}{2 - \dfrac{n-1}{n}} = \frac{1}{\dfrac{n+1}{n}} = \frac{n}{n+1}$$

The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :

$\forall n \geqslant 1,\ u_n = \dfrac{n-1}{n}$.

We then obtain : $u_{2014} = \dfrac{2013}{2014}$

Exercise 12

Basis step : $f_1(x) = x$ then $f_1'(x) = 1 = 1x^0$. The basis step is established.

Induction step : Supposing that $f_n'(x) = nx^{n-1}$ show that $f_{n+1}'(x) = (n+1)x^n$

$f_{n+1}(x) = x^{n+1} = x \times x^n$ derivative of the product of two functions :

$$f_{n+1}'(x) = x^n + x \times nx^{n-1} = x^n + nx^n = (n+1)x^n$$

The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :

$\forall n \in \mathbb{N}^*, \ f_n'(x) = nx^{n-1}$

Exercise 13

1) a) $u_1 = \left(1 + \dfrac{2}{1}\right) \times 5 + \dfrac{6}{1} = 21\,;\quad u_2 = \left(1 + \dfrac{2}{2}\right) \times 21 + \dfrac{6}{2} = 45$

$u_3 = \left(1 + \dfrac{2}{3}\right) \times 45 + \dfrac{6}{3} = 77$

b) We can write the following algorithm :

Variables: N, I integers
U, V, D real numbers
Inputs and initialization
Read N
$5 \to U$
Processing
for I from 1 to N **do**
$\left(1 + \dfrac{2}{I}\right) U + \dfrac{6}{I} \to V$
$V - U \to D$
$V \to U$
end
Output : Print U, D

⚠ To calculate d_{n-1}, we need u_n and u_{n-1} hence the use of the variable V. For $N = 1$ we obtain u_1 and d_0.

n	0	1	2	3	4	5	6
u_n	5	21	45	77	117	165	221
d_n	16	24	32	40	48	56	

We can conjecture that (d_n) is an arithmetic sequence with a first term of $d_0 = 16$ with a common difference of $r = 8$ which is verified from d_0 to d_5.

2) If (v_n) is arithmetic with a common difference of 8 and a first term of $v_0 = 16$, we then have : $v_n = 16 + 8n$

The sum S of n first terms is :

$$\begin{aligned} S &= v_0 + v_1 + v_2 + \cdots + v_{n-1} \\ &= n \times \frac{v_0 + v_{n-1}}{2} = n \times \frac{16 + 16 + 8(n-1)}{2} \\ &= n \times \frac{24 + 8n}{2} = 4n^2 + 12n \end{aligned}$$

3) **Basis step :** $4 \times 0^2 + 12 \times 0 + 5 = 5 = u_0$. The basis step is established.

Induction step : Supposing that $u_n = 4n^2 + 12n + 5$
show that $u_{n+1} = 4(n+1)^2 + 12(n+1) + 5$.

By expanding the expression : $4(n+1)^2 + 12(n+1) + 5 = 4n^2 + 20n + 21$

$$\begin{aligned} u_{n+1} &= \left(1 + \frac{2}{n+1}\right) u_n + \frac{6}{n+1} \\ &= u_n + \frac{2u_n + 6}{n+1} = u_n + \frac{8n^2 + 24n + 10 + 6}{n+1} \\ &= u_n + 8 \times \frac{n^2 + 3n + 2}{n+1} = u_n + 8 \times \frac{(n+1)(n+2)}{n+1} \\ &= u_n + 8(n+2) = 4n^2 + 12n + 5 + 8n + 16 \\ &= 4n^2 + 20n + 21 \end{aligned}$$

because $n^2 + 3n + 2 = (n+1)(n+2)$

The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :
$\forall n \in \mathbb{N}, \;\; u_n = 4n^2 + 12n + 5$

4) $d_n = u_{n+1} - u_n = 4n^2 + 20n + 21 - 4n^2 - 12n - 5 = 16 + 8n$
so (d_n) is an arithmetic sequence with a first term of 16 and a common difference of 8.

Exercise 14

1) $u_n = \dfrac{n\left(2 + \dfrac{5}{n}\right)}{n\left(3 - \dfrac{2}{n}\right)} = \dfrac{\left(2 + \dfrac{5}{n}\right)}{\left(3 - \dfrac{2}{n}\right)} \qquad \text{as } \lim\limits_{n\to+\infty} \dfrac{5}{n} = \lim\limits_{n\to+\infty} \dfrac{2}{n} = 0$

Using the sum, product and quotient of limits : $\lim\limits_{n\to+\infty} u_n = \dfrac{2}{3}$

2) $u_n = \dfrac{n}{4} - 2 + \dfrac{2n}{n^2\left(1 + \dfrac{5}{n^2}\right)} = \dfrac{n}{4} - 2 + \dfrac{2}{n\left(1 + \dfrac{5}{n^2}\right)}$

as $\lim\limits_{n\to+\infty} \dfrac{n}{4} - 2 = +\infty$ and $\lim\limits_{n\to+\infty} 1 + \dfrac{5}{n^2} = 1$

Using the sum, product and quotient of limits : $\lim\limits_{n\to+\infty} u_n = +\infty$

3) $u_n = \dfrac{n^2\left(-3 + \dfrac{2}{n} + \dfrac{1}{n^2}\right)}{2n^2\left(1 + \dfrac{2}{n} + \dfrac{1}{n^2}\right)} = \dfrac{-3 + \dfrac{2}{n} + \dfrac{1}{n^2}}{2\left(1 + \dfrac{2}{n} + \dfrac{1}{n^2}\right)} \qquad \text{as } \lim\limits_{n\to+\infty} \dfrac{2}{n} + \dfrac{1}{n^2} = 0$

Using the sum, product and quotient of limits : $\lim\limits_{n\to+\infty} u_n = -\dfrac{3}{2}$

Exercise 15

1) $u_n = \dfrac{n\left(10 - \dfrac{3}{n}\right)}{n^2\left(1 - \dfrac{2}{n^2}\right)} = \dfrac{10 - \dfrac{3}{n}}{n\left(1 - \dfrac{2}{n^2}\right)} \qquad \text{as } \lim\limits_{n\to+\infty} -\dfrac{3}{n} = \lim\limits_{n\to+\infty} -\dfrac{2}{n^2} = 0$

Using the sum, product and quotient of limits : $\lim\limits_{n\to+\infty} u_n = 0$

2) $u_n = \dfrac{n^2\left(2 - \dfrac{1}{n^2}\right)}{n\left(3 + \dfrac{2}{n}\right)} = \dfrac{n\left(2 - \dfrac{1}{n^2}\right)}{3 + \dfrac{2}{n}} \qquad \text{as } \lim\limits_{n\to+\infty} -\dfrac{1}{n^2} = \lim\limits_{n\to+\infty} \dfrac{2}{n} = 0$

Using the sum, product and quotient of limits : $\lim\limits_{n\to+\infty} u_n = +\infty$

3) $u_n = \dfrac{3n^2-4-3n^2-3n}{n+1} = \dfrac{n\left(-3-\dfrac{4}{n}\right)}{n\left(1+\dfrac{1}{n}\right)} = \dfrac{-3-\dfrac{4}{n}}{1+\dfrac{1}{n}}$

as $\lim\limits_{n\to+\infty} -\dfrac{4}{n} = \lim\limits_{n\to+\infty} \dfrac{1}{n} = 0.$

Using the sum, product and quotient of limits : $\lim\limits_{n\to+\infty} u_n = -3$

Exercise 16

1) $u_n = \dfrac{1}{\sqrt{n+2}}$ as $\lim\limits_{n\to+\infty} n+2 = +\infty$ and $\lim\limits_{n\to+\infty} \sqrt{n} = +\infty$

Using composition and quotient of limits : $\lim\limits_{n\to+\infty} u_n = 0$

2) $u_n = \dfrac{n(\sqrt{n}+1)}{n\left(1+\dfrac{1}{n}\right)} = \dfrac{\sqrt{n}+1}{1-\dfrac{2}{n}}$ as $\lim \sqrt{n}+1 = +\infty$ and $\lim\limits_{n\to+\infty} 1-\dfrac{2}{n} = 1$

The quotient is : $\lim\limits_{n\to+\infty} u_n = +\infty$

Exercise 17

a) $\forall n \in \mathbb{N}^*,\ -\dfrac{1}{\sqrt{n}} \leqslant u_n \leqslant \dfrac{1}{\sqrt{n}}$ as $\lim\limits_{n\to+\infty} -\dfrac{1}{\sqrt{n}} = \lim\limits_{n\to+\infty} \dfrac{1}{\sqrt{n}} = 0$

By virtue of the squeeze theorem : $\lim\limits_{n\to+\infty} u_n = 0$

b) $\forall n \in \mathbb{N},\ v_n \geqslant n$ as $\lim\limits_{n\to+\infty} n = +\infty$. By comparison of limits : $\lim\limits_{n\to+\infty} u_n = +\infty$

Exercise 18

a) $u_1 = \dfrac{1}{2}$, $u_2 = \dfrac{11}{15}$, $u_3 = \dfrac{181}{220}$

b) $u_{10} = 0{,}948$; $u_{20} = 0{,}974$,

$u_{50} = 0{,}990$

Thus, we can conjecture that the sequence converges to 1

Variables: N, I integers U : real numbers
Inputs and initialization
  Read N
  $0 \to U$
Processing
  for I from 1 to N **do**
    $U + \dfrac{N}{N^2+I} \to U$
  end
Output : Print U

c) We bounded above all the terms of u_n by $\dfrac{n}{n^2+1}$ and bounded below by $\dfrac{n}{n^2+n}$

d) $\dfrac{n^2}{n^2+n} = \dfrac{1}{1+\dfrac{1}{n}}$ and $\dfrac{n^2}{n^2+1} = \dfrac{1}{1+\dfrac{1}{n^2}}$ as $\lim\limits_{n\to+\infty} \dfrac{1}{1+\dfrac{1}{n}} = \lim\limits_{n\to+\infty} \dfrac{1}{1+\dfrac{1}{n^2}} = 1$

by virtue of the squeeze theorem : $\lim\limits_{n\to+\infty} u_n = 1$

Exercise 19

(u_n) is the sum of the $n+1$ first terms of a geometric sequence with a common ratio of $\dfrac{1}{2}$ and a 1st term of 1.

$$u_n = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2\left(1-\left(\frac{1}{2}\right)^{n+1}\right)$$

as $\lim\limits_{n\to+\infty} \left(\dfrac{1}{2}\right)^{n+1} = 0$ because $-1 < \dfrac{1}{2} < 1.$

Using the sum and product of limits : $\lim\limits_{n\to+\infty} u_n = 2$

Exercise 20

1) a) $v_{n+1} = u_{n+1} + 3 = \dfrac{1}{3}u_n - 2 + 3 = \dfrac{1}{3}(u_n+3) = \dfrac{1}{3}v_n$

$\forall n \in \mathbb{N},\ \dfrac{v_{n+1}}{v_n} = \dfrac{1}{3}$. So the sequence (v_n) is geometric with a common ratio of $q = \dfrac{1}{3}$ and a 1st term of $v_0 = u_0 + 3 = 6$

b) $v_n = v_0 q^n = 6\left(\dfrac{1}{3}\right)^n$ and $u_n = 6\left(\dfrac{1}{3}\right)^n + 3$

2) a) S_n is the sum of $n+1$ first terms of a geometric sequence :

$$S_n = 6 \times \frac{1-\left(\frac{1}{3}\right)^{n+1}}{1-\frac{1}{3}} = 9\left(1-\left(\frac{1}{3}\right)^{n+1}\right)$$

$$\begin{aligned} S'_n &= u_0 + u_1 + \cdots + u_n \\ &= (v_0+3) + (v_1+3) + \cdots + (v_n+3) \\ &= S_n + 3(n+1) \end{aligned}$$

b) As $\lim\limits_{n\to+\infty} \left(\dfrac{1}{3}\right)^{n+1} = 0$ because $-1 < \dfrac{1}{3} < 1.$

Using the sum and product of limits : $\lim\limits_{n\to+\infty} S_n = 9$

$\lim\limits_{n\to+\infty} 3(n+1) = +\infty.$ The sum is $\lim\limits_{n\to+\infty} S'_n = -\infty$

Exercise 21

a) $v_{n+1} = (n+1)u_{n+1} - 1 = (n+1) \times \dfrac{nu_n + 1}{2(n+1)} - 1 = \dfrac{nu_n + 1 - 2}{2}$

$= \dfrac{1}{2}(nu_n - 1) = \dfrac{1}{2}v_n$

$\forall n \in \mathbb{N},\ \dfrac{v_{n+1}}{v_n} = 0,5,$ the sequence (v_n) is geometric with a common ratio of $q = 0,5$ and a 1st term of $v_0 = -1$

b) $v_n = v_0 q^n = -0,5^n$ so $u_n = \dfrac{v_n + 1}{n} = \dfrac{1 + 0,5^n}{n}$

c) $\lim\limits_{n\to+\infty} 0,5^n = 0$ because $-1 < 0,5 < 1$
so the sum is $\lim\limits_{n\to+\infty} 1 + 0,5^n = 1$ and the quotient is $\lim\limits_{n\to+\infty} u_n = 0$

d) $u_{n+1} - u_n = \dfrac{1+0,5^{n+1}}{n+1} - \dfrac{1+0,5^n}{n} = \dfrac{n + 0,5^n \times 0,5n - n - 1 - 0,5^n \times n - 0,5^n}{n(n+1)}$

$$= \frac{-0.5^n \times 0,5n - 1 - 0,5^n}{n(n+1)} = -\frac{1+0,5^n(1+0,5n)}{n(n+1)}$$

$\forall n \in \mathbb{N},\ u_{n+1} - u_n \leqslant 0$ then the sequence (u_n) is decreasing.

Exercise 22

a) (u_n) is bounded by $[-1;1]$ because $\forall n \in \mathbb{N},\ -1 \leqslant \sin n \leqslant 1$

b) (u_n) is bounded above by 1 because
$\forall n \in \mathbb{N},\ 1 + n^2 \geqslant 1 \ \Rightarrow\ \dfrac{1}{1+n^2} \leqslant 1$ (the inverse function is decreasing)

c) (u_n) is bounded below by 1 because $\forall n \in \mathbb{N},\ 2^n \geqslant 1$

d) (u_n) is bounded below by -1 because $\forall n \in \mathbb{N},\ n + \cos n \geqslant -1$

e) (u_n) is bounded neither above nor below.

Exercise 23

a) $\forall n \in \mathbb{N},\ u_{n+1} - u_n = 2n + 3 > 0$, the sequence (u_n) is increasing.

b) **Basis step :** $u_0 = 1 > 0^2$. The basis step is established.

Induction step : Supposing that $u_n > n^2$ show that $u_{n+1} > (n+1)^2$

$u_{n+1} = u_n + 2n + 3 > n^2 + 2n + 1$ then $u_{n+1} > (n+1)^2$

The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :
$\forall n \in \mathbb{N},\ u_n > n^2$

c) $\lim\limits_{n\to+\infty} n^2 = +\infty$ by comparison of limits $\lim\limits_{n\to+\infty} u_n = +\infty$

Exercise 24

a) **False**, not necessary.

Counterexample : $u_n = (-1)^n \times n$
(u_n) is clearly bounded neither above nor below. There are alternating large values positive values and large negative values.

b) **False**, a sequence can move towards a finite limit and be increasing.

Counterexample : $u_n = 1 - 0,5^n$ the sequence $0,5^n$ is decreasing hence $(u_n)$is increasing.
As $\lim\limits_{n\to+\infty} 0,5^n = 0$ car $-1 < 0,5 < 1$ then $\lim\limits_{n\to+\infty} u_n = 1$

c) **True**. If $\lim\limits_{n \to +\infty} u_n = +\infty$ this means any interval $]A; +\infty[$ contains all the terms of the sequence from a certain index, however large A may be.

d) **False**. A sequence can approach infinity without being increasing.

Counterexample : $u_n = n + (-1)^n$. So :
$u_{n+1} - u_n = 1 - 2(-1)^n$ has alternating values -1 and 3. The sequence is neither increasing nor decreasing.
$\forall n \in \mathbb{N}, \ u_n \geqslant n - 1$ by comparison of limits $\lim\limits_{n \to +\infty} u_n = +\infty$

EXERCISE 25

Part A

1) a) **Basis step :** $u_0 = 0$ then $0 \leqslant u_0 < 1$. The basis step is established.

Induction step : Supposing that $0 \leqslant u_n < 1$, show that $0 \leqslant u_n < 1$.

$$u_{n+1} = \frac{2u_n + 1}{u_n + 2} = \frac{2(u_n+2) - 3}{u_n + 2} = 2 - \frac{3}{u_n + 2}$$

$$0 \leqslant u_n < 1 \Leftrightarrow 2 \leqslant u_n + 2 < 3 \Leftrightarrow \frac{1}{3} < \frac{1}{u_n + 2} \leqslant \frac{1}{2} \Leftrightarrow$$
$$-\frac{3}{2} \leqslant -\frac{3}{u_n + 2} < -1 \Leftrightarrow 0 \leqslant \frac{1}{2} \leqslant 2 - \frac{3}{u_n + 2} < 1$$

Hence $0 \leqslant u_{n+1} < 1$. The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :
$\forall n \in \mathbb{N}, \ 0 \leqslant u_n < 1$

b) $$u_{n+1} - u_n = \frac{2u_n + 1}{u_n + 2} - u_n = \frac{2u_n + 1 - u_n^2 - 2u_n}{u_n + 2} = \frac{1 - u_n^2}{u_n + 2}$$

We have $\forall n \in \mathbb{N}$,
$0 \leqslant u_n < 1 \Rightarrow 0 \leqslant u_n^2 < 1 \Rightarrow -1 < -u_n^2 \leqslant 0 \Rightarrow 1 - u_n^2 > 0$

$0 \leqslant u_n < 1 \Rightarrow u_n + 2 > 2$

then $\forall n \in \mathbb{N}, \ u_{n+1} - u_n > 0$. The sequence (u_n) is increasing.

2) The sequence (u_n) is increasing and bounded above by 1, then by virtue of the theorem of monotonic sequences, the sequence (u_n) converges to ℓ such that $0 \leqslant \ell \leqslant 1$

3) a) $$\ell = \frac{2\ell + 1}{\ell + 2} \Leftrightarrow \ell^2 + 2\ell = 2\ell + 1 \Leftrightarrow \ell^2 = 1$$

As ℓ is positive $\ell = 1$

b) The algorithm shown opposite can be written to find that $N = 7$

Variables: N : integer U : real number
Inputs and initialization
  $0 \to N$
  $0 \to U$
Processing
  while $|U-1| \geqslant 10^{-3}$ **do**
    $\dfrac{2U+1}{U+2} \to U$
    $N+1 \to N$
  end
Output : Print N

Part B

1) $v_{n+1} = \dfrac{u_{n+1}-1}{u_{n+1}+1} = \dfrac{2u_n+1-u_n-2}{u_n+2} \times \dfrac{u_n+2}{2u_n+1+u_n+2} = \dfrac{u_n-1}{3u_n+3} = \dfrac{1}{3}v_n$

$\forall n \in \mathbb{N},\ \dfrac{v_{n+1}}{v_n} = \dfrac{1}{3}$, the sequence (v_n) is geometric with a common ratio of $q = \dfrac{1}{3}$ and a 1st term of $v_0 = -1$

2) $v_n = v_n q^n = -\left(\dfrac{1}{3}\right)^n$. We express u_n in terms of v_n :

$$v_n = \frac{u_n-1}{u_n+1} \Leftrightarrow v_n u_n + v_n = u_n - 1 \Leftrightarrow u_n(v_n-1) = -v_n - 1 \Leftrightarrow$$

$$u_n = \frac{1+v_n}{1-v_n} \Leftrightarrow u_n = \frac{1-\left(\dfrac{1}{3}\right)^n}{1+\left(\dfrac{1}{3}\right)^n}$$

3) $\lim\limits_{n\to+\infty} \left(\dfrac{1}{3}\right)^n = 0$ because $-1 < \dfrac{1}{3} < 1$. The sums and quotient are, $\lim\limits_{n\to+\infty} u_n = 1$

Exercise 26

1) a) We find $u = 1,8340$ to within 10^{-4}.

b) This algorithm calculates u_n, n given.

c) We have the following table :

n	1	5	10	15	20
u_n	1,4142	1,9571	1,9986	$\simeq 2$	$\simeq 2$

We can conjecture that the sequence (u_n) is increasing and converges to 2

2) a) By induction.

Basis step : $u_0 = 1 \Rightarrow 0 < u_0 \leqslant 2$. The basis step is established.

Induction step : Supposing that $0 < u_n \leqslant 2$ show that $0 < u_{n+1} \leqslant 2$

$0 < u_n \leqslant 2 \Rightarrow 0 < 2u_n \leqslant 4 \Rightarrow 0 < \sqrt{2u_n} \leqslant 2 \Rightarrow 0 < u_{n+1} \leqslant 2$
because the square root function is increasing on $\mathbb{R}_+$. The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :
$\forall n \in \mathbb{N},\ 0 < u_n \leqslant 2$

b) $\forall n \in \mathbb{N},\ u_n > 0 : \dfrac{u_{n+1}}{u_n} = \dfrac{\sqrt{2u_n}}{u_n} = \sqrt{\dfrac{2}{u_n}}$

$$0 < u_n \leqslant 2 \quad \Rightarrow \quad \frac{1}{u_n} \geqslant \frac{1}{2} \quad \Rightarrow \quad \frac{2}{u_n} \geqslant 1 \quad \Rightarrow \quad \sqrt{\frac{2}{u_n}} \geqslant 1$$

The sequence (u_n) is increasing.

c) The sequence (u_n) is increasing and bounded below by 2, by virtue of the theorem of monotonic sequences, the sequence (u_n) is convergent.

EXERCISE 27

a) True, because $\forall n \in \mathbb{N}^*,\ -1 \leqslant u_n \leqslant 1$

b) False because the sequence (u_n) alternately takes the values -1 and 1.

c) True, $\forall n \in \mathbb{N}^*,\ -\dfrac{1}{n} \leqslant \dfrac{u_n}{n} \leqslant \dfrac{1}{n}$, as $\lim\limits_{n\to+\infty} -\dfrac{1}{n} = \lim\limits_{n\to+\infty} \dfrac{1}{n} = 0$

according to the squeeze theorem, the sequence converges to 0.

d) False, we can only say that the sequence converges to a positive number or to zero. Counterexample : Consider $v_n = 1 + \dfrac{1}{n}$

As the inverse function is decreasing on $]0;+\infty[$, the sequence (v_n) is decreasing. But $\lim\limits_{n\to+\infty} v_n = 1$

EXERCISE 28

1) a) $u_1 = \dfrac{7}{3} \simeq 2,33$; $u_2 = \dfrac{26}{9} \simeq 2,89$; $u_3 = \dfrac{97}{27} \simeq 3,59$; $u_4 = \dfrac{356}{81} \simeq 4,40$

b) We can conjecture that the sequence is increasing

2) a) By induction

Basis step : $u_0 = 2$ and $0+3 = 3$ so $u_0 \leqslant 0+3$. The basis step is established.

Induction step : Supposing that $u_n \leqslant n+3$, show that $u_{n+1} \leqslant n+4$

$$u \leqslant n+3 \quad \Leftrightarrow \quad \frac{2}{3}u_n \leqslant \frac{2}{3}(n+3) \quad \Leftrightarrow \quad \frac{2}{3}u_n + \frac{1}{3}n + 1 \leqslant \frac{2}{3}n + 2 + \frac{1}{3}n + 1$$

$$u_{n+1} \leqslant n+3 \leqslant n+4$$

The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :

$\forall n \in \mathbb{N},\ u_n \leqslant n+3$

b) $u_{n+1} - u_n = \dfrac{2}{3}u_n + \dfrac{1}{3}n + 1 - u_n = \dfrac{1}{3}(n+3-u_n)$

c) As $u_n \leqslant n+3$ then $n+3-u_n > 0$ so $u_{n+1} - u_n > 0$. The sequence is increasing.

3) a) $v_{n+1} = u_{n+1} - (n+1) = \dfrac{2}{3}u_n + \dfrac{1}{3}n + 1 - n - 1 = \dfrac{2}{3}u_n - \dfrac{2}{3}n$

$$= \frac{2}{3}(u_n - n) = \frac{2}{3}v_n$$

$\forall n \in \mathbb{N},\ \dfrac{v_{n+1}}{v_n} = \dfrac{2}{3}$. The sequence (v_n) is geometric with a common ratio of $q = \dfrac{2}{3}$ and a 1st term of $v_0 = 2$.

b) $v_n = v_0\, q^n = 2\left(\dfrac{2}{3}\right)^n$ then $u_n = 2\left(\dfrac{2}{3}\right)^n + n$

c) $\lim\limits_{n\to+\infty} \left(\dfrac{2}{3}\right)^n = 0$ car $-1 < \dfrac{2}{3} < 1$. By sum : $\lim\limits_{n\to+\infty} u_n = +\infty$

4) a)
$$\begin{aligned} S_n &= u_0 + u_1 + u_2 + \cdots + u_n \\ &= v_0 + (v_1 + 1) + (v_2 + 2) + \cdots + (v_n + n) \\ &= (v_0 + v_1 + v_2 \cdots + v_n) + 1 + 2 + \cdots + n \\ &= v_0 \times \frac{1 - q^{n+1}}{1 - \frac{2}{3}} + \frac{n(n+1)}{2} \\ &= 6\left(1 + \left(\frac{2}{3}\right)^n\right) + \frac{n(n+1)}{2} \end{aligned}$$

b) $T_n = \dfrac{S_n}{n^2} = \dfrac{6}{n^2}\left(1 + \left(\dfrac{2}{3}\right)^n\right) + \dfrac{1}{2} + \dfrac{1}{2n}$

$\lim\limits_{n\to+\infty} \left(\dfrac{2}{3}\right)^n = 0\,;\ \lim\limits_{n\to+\infty} \dfrac{6}{n^2} = 0\,;\ \lim\limits_{n\to+\infty} \dfrac{1}{2n} = 0$

Using the sum and product of limits $\lim\limits_{n\to+\infty} T_n = \dfrac{1}{2}$

Exercise 29

Part A

1) The following algorithm can be written :

Variables: N, I : integers V : real number
Inputs and initialization
  Read N
  $1 \to V$
Processing and outputs
  for I from 1 to N **do**
    $\dfrac{9}{6-V} \to V$
    Print V
  end

2)

n	3	4	5	6	7	8
u_n	2,333	2,455	2,538	2,600	2,647	2,684

The sequence (v_n) is increasing and seems to converge to 3

3) a) **Basis step :** $v_0 = 1$ then $0 < v_0 < 3$. The basis step is established.

Induction step : Supposing that $0 < u_n < 3$ show that $0 < u_{n+1} < 3$

$$0 < v_n < 3 \Leftrightarrow -3 < -v_n < 0 \Leftrightarrow 3 < 6 - v_n < 6 \Leftrightarrow \frac{1}{6} < \frac{1}{6 - v_n} < \frac{1}{3} \Leftrightarrow$$
$$\frac{3}{2} < \frac{9}{6 - v_n} < 3 \Leftrightarrow 0 < u_{n+1} < 3$$

The induction step is established.
Therefore, from the basis and induction steps, we can conclude that :
$\forall n \in \mathbb{N},\ 0 < u_n < 3$

b) $$v_{n+1} - v_n = \frac{9}{6 - v_n} - v_n = \frac{9 - 6v_n + v_n^2}{6 - v_n} = \frac{(3 - v_n)^2}{6 - v_n}$$

As $6 - v_n > 0$ and $3 - v_n > 0$

$\forall n \in \mathbb{N},\ v_{n+1} - v_n > 0$. The sequence (v_n) is increasing.

c) The sequence (v_n) is increasing and bounded below by 3, by the theorem of monotonic sequences, the sequence (v_n) is convergent.

Part B

1) $$w_{n+1} = \frac{1}{v_{n+1} - 3} = \frac{1}{\dfrac{9}{6 - v_n} - 3} = \frac{6 - v_n}{9 - 18 + 3v_n} = \frac{6 - v_n}{3(v_n - 3)}$$

$$w_{n+1} - w_n = \frac{6 - v_n}{3(v_n - 3)} - \frac{1}{v_n - 3} = \frac{6 - v_n - 3}{3(v_n - 3)} = \frac{-(v_n - 3)}{3(v_n - 3)} = -\frac{1}{3}$$

$\forall n \in \mathbb{N},\ w_{n+1} - w_n = -\dfrac{1}{3}$, the sequence (w_n) is arithmetic with a common difference of $r = -\dfrac{1}{3}$ and a 1st term of $w_0 = -\dfrac{1}{2}$

2) $w_n = w_0 + nr = -\dfrac{1}{2} - \dfrac{n}{3}$ from $w_n = \dfrac{1}{v_n - 3}$ we have $v_n = \dfrac{1 + 3w_n}{w_n}$

$$v_n = \frac{1 - \dfrac{3}{2} - n}{-\dfrac{1}{2} - \dfrac{n}{3}} = \frac{-3 - 6n}{-3 - 2n} = \frac{6n + 3}{2n + 3}$$

3) $$v_n = \frac{n\left(6 + \dfrac{3}{n}\right)}{n\left(2 + \dfrac{3}{n}\right)} = \frac{6 + \dfrac{3}{n}}{2 + \dfrac{3}{n}} \quad \text{as} \quad \lim_{n \to +\infty} \frac{3}{n} = 0$$

the sum and quotient are $\displaystyle\lim_{n \to +\infty} v_n = \frac{6}{2} = 3$

Exercise 30

1) $u_2 = \dfrac{3}{8}$; $u_3 = \dfrac{1}{4}$; $u_4 = \dfrac{5}{32}$

2) a) **Basis step :** $u_1 = \dfrac{1}{2} > 0$. The basis step is established.

Induction step : Supposing that $u_n > 0$, show that $u_{n+1} > 0$

$u_{n+1} = \dfrac{n+1}{2n} u_n > 0$ because $n \geqslant 1$
The induction step is established.
Therefore, from the basis and induction steps, we can conclude that :
$\forall n \geqslant 1, \quad u_n > 0$

b) $\dfrac{u_{n+1}}{u_n} = \dfrac{n+1}{2n}$. As $n \geqslant 1 \Leftrightarrow 2n \geqslant n+1$.

then $\forall n \geqslant 1, \ \dfrac{u_{n+1}}{u_n} \leqslant 1$. The sequence (u_n) is decreasing.

c) The sequence (u_n) is decreasing and bounded below by 0, by virtue of the theorem of monotonic sequences, the sequence (u_n) is convergent.

Exercise 31

Part A

1) **Basis step :** $u_0 = 2 > 1$. The basis step is established.

Induction step : Supposing that $u_n > 1$, show that $u_{n+1} > 1$

$$u_{n+1} = \frac{1+3u_n}{3+u_n} = \frac{3+u_n+2u_n-2}{3+u_n} = 1 + \frac{2(u_n-1)}{3+u_n}$$

$u_n > 1 \Rightarrow u_n - 1 > 0$ et $3+u_n > 0$

We deduce that $\dfrac{2(u_n-1)}{3+u_n} > 0 \Rightarrow u_{n+1} > 1$

The induction step is established.

Therefore, from the basis and induction steps, we can conclude that :
$\forall n \in \mathbb{N}, \quad u_n > 1$

2) a) $$u_{n+1} - u_n = \frac{1+3u_n}{3+u_n} - u_n = \frac{1+3u_n-3u_n-u_n^2}{3+u_n} = \frac{1-u_n^2}{3-u_n} = \frac{(1-u_n)(1+u_n)}{3+u_n}$$

b) As $u_n > 1$ then $1-u_n < 0$; $1+u_n > 0$ and $3+u_n > 0$

$\forall n \in \mathbb{N}, \quad u_{n+1} - u_n < 0$. The sequence (u_n) is decreasing.

The sequence (u_n) is decreasing and bounded below by 1, according to the theorem of monotonic sequences, the sequence (u_n) is convergent.

Part B

1) We have the following table :

i	1	2	3	4	5	6	7	8	9
u	0,8000	1,0769	0,9756	1,0083	0,9973	1,0009	0,9997	1,0001	$\simeq 1$

2) a) $v_{n+1} = \dfrac{u_{n+1}-1}{u_{n+1}+1} = \dfrac{\dfrac{1+0,5u_n}{0,5+u_n}-1}{\dfrac{1+0,5u_n}{0,5+u_n}+1} = \dfrac{1+0,5u_n-0,5-u_n}{1+0,5u_n+0,5+u_n}$

$= \dfrac{0,5-0,5u_n}{1,5+1,5u_n} = -\dfrac{0,5}{1,5} \times \dfrac{u_n-1}{u_n+1} = -\dfrac{1}{3}v_n$

$\forall n \in \mathbb{N}$, $\dfrac{v_{n+1}}{v_n} = -\dfrac{1}{3}$, the sequence (v_n) is geometric with a common ratio of $q = -\dfrac{1}{3}$ and a 1st term of $v_0 = \dfrac{1}{3}$

b) $v_n = v_0 q^n = \dfrac{1}{3} \times \left(-\dfrac{1}{3}\right)^n$

c) $\forall n \in \mathbb{N}$, $\left(-\dfrac{1}{3}\right)^n \leqslant 1 \quad \Rightarrow \quad v_n \leqslant \dfrac{1}{3}$

d) $v_n = \dfrac{u_n-1}{u_n+1} \quad \Leftrightarrow \quad v_n u_n + v_n = u_n - 1 \quad \Leftrightarrow u_n(v_n-1) = -v_n - 1$

$\Leftrightarrow \quad u_n = \dfrac{1+v_n}{1-v_n}$

e) $\displaystyle\lim_{n\to+\infty} \left(-\frac{1}{3}\right)^n = 0$ because $-1 < -\dfrac{1}{3} < 1$

The product is $\displaystyle\lim_{n\to+\infty} v_n = 0$, using the sum and quotient of limits $\displaystyle\lim_{n\to+\infty} u_n = 1$

EXERCISE 32

1) a) We have the following table

n	0	1	2	3	4	5	6	7	8
u_n	2	3,4	2,18	1,19	0,61	0,31	0,16	0,08	0,04

b) Conjecture : The sequence is decreasing and converges to 0.

2) a) **Basis step : :** $u_1 = \dfrac{2}{5} + 3 \times 0,5^0 = 3,4 \; \dfrac{15}{4} \times 0,5 = 1,875$ then $u_1 \geqslant 1,875$. The basis step is established.

Supposing that $u_n \geqslant \dfrac{15}{4} \times 0,5^n$, show that $u_{n+1} \geqslant \dfrac{15}{4} \times 0,5^{n+1}$

$u_n \geqslant \dfrac{15}{4} \times 0,5^n \quad \Leftrightarrow \quad \dfrac{1}{5}u_n \geqslant \dfrac{3}{4} \times 0,5^n \quad \Leftrightarrow \dfrac{1}{5}u_n + \times 0,5^n \geqslant \dfrac{3}{4} \times 0,5^2 + 3 \times 0,5^n$

$\Leftrightarrow \quad u_{n+1} \geqslant \left(\dfrac{3}{4} + 3\right) \times 0,5^n \quad \Leftrightarrow \quad u_{n+1} \geqslant \dfrac{15}{4} \times 0,5^n$

as $\forall n \in \mathbb{N}$, $0,5^n \geqslant 0,5^{n+1}$ then $u_{n+1} \geqslant \dfrac{15}{4} \times 0,5^{n+1}$

The induction step is established.
Therefore, from the basis and induction steps, we can conclude that :
$\forall \in \mathbb{N}^*$, $u_n \geqslant \dfrac{15}{4} \times 0,5^n$

b) $u_{n+1} - u_n = \frac{1}{5}u_n + 3 \times 0,5^n - u_n = -\frac{4}{5}u_n + 3 \times 0,5^n$

$u_n \geqslant \frac{15}{4} \times 0,5^n \quad \Leftrightarrow \quad -\frac{4}{5}u_n \leqslant -\frac{4}{5} \times \frac{15}{4} \times 0,5^n \quad \Leftrightarrow \quad -\frac{4}{5}u_n \leqslant -3 \times 0,5^n$

We then deduce : $u_{n+1} - u_n \leqslant 0$

The sequence $u_n)$ is decreasing from index 1.

c) $\forall n \in \mathbb{N}^*, \quad u_n \geqslant \frac{15}{4} \times 0,5^n \geqslant 0$

The sequence (u_n) is decreasing and bounded below by 0 from index 1, by using the theorem of monotonic sequences, we can conclude that the sequence (u_n) is convergent.

3) $v_{n+1} = u_{n+1} - 10 \times 0,5^{n+1} = \frac{1}{5}u_n + 3 \times 0,5^n - 5 \times 0,5^n$

$$= \frac{1}{5}u_n - 2 \times 0,5^n = \frac{1}{5}\left(u_n - 10 \times 0,5^n\right) = \frac{1}{5}v_n$$

$\forall n \in \mathbb{N}, \quad \frac{v_{n+1}}{v_n} = \frac{1}{5}$, the sequence (v_n) is geometric with a common ratio of $q = \frac{1}{5}$ and a 1st term of $v_n = u_0 - 10 \times 0,5^0 = -8$

$v_n = v_n q^n = -8 \times \left(\frac{1}{5}\right)^n$ so $u_n = v_n + 10 \times 0,5^n = -8\left(\frac{1}{5}\right)^n + 10 \times 0,5^n$

b) $\lim\limits_{n \to +\infty} \left(\frac{1}{5}\right)^n = 0$ and $\lim\limits_{n \to +\infty} 0,5^n = 0$ because $-1 < \frac{1}{5} < 0,5 < 1$

Using the product and sum of limits $\lim\limits_{n \to +\infty} u_n = 0$

4) Consider the following algorithm :

Variables: n : integer u : real number
Inputs and initialization
  $0 \to n$
  $2 \to u$
Processing
  while $u > 0,01$ **do**
    $n + 1 \to n$
    $\frac{1}{5}u + 3 \times 0,5^{n-1} \to u$ *
  end
Output : Print n

* : As n is incremented before the new value of u is calculated, $n - 1$ is used instead of n.

So the final result is $n = 10$