Limits of functions

Proofreading of English by Laurence Weinstock

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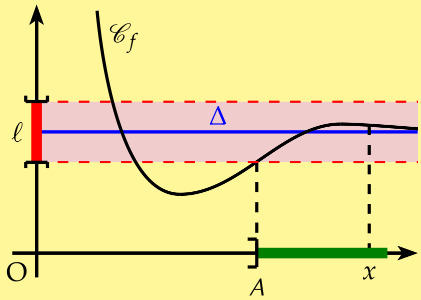
1 Limits at infinity

1.1 Finite limit at infinity

Definition 1 : A function f is said to have a finite limit ℓ at $+\infty$, if any open interval containing ℓ, contains all values of $f(x)$ for x large enough - that is to say, for x in an interval $]A;+\infty[$. It is written :

$$\lim_{x\to+\infty} f(x) = \ell$$



The line Δ defined by $y = \ell$ is the **horizontal asymptote** of the graph $\mathscr{C}_f$

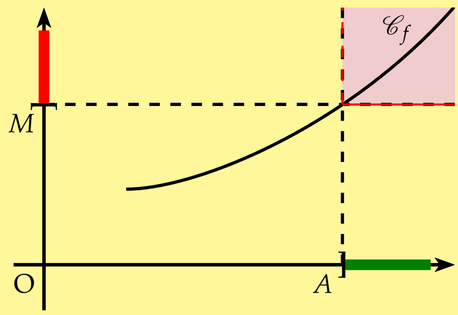
Note : A finite limit at $-\infty$ is likewise defined and is denoted $\lim_{x\to-\infty} f(x) = \ell$.

Example : The common functions : $x \mapsto \dfrac{1}{x}$, $x \mapsto \dfrac{1}{x^n}$ and $x \mapsto \dfrac{1}{\sqrt{x}}$ have a limit of 0 at $+\infty$. The first two also have a limit of 0 at $-\infty$. Their horizontal asymptote is the x-axis.

1.2 Infinite limit at infinity

Definition 2 : A function f is said to have an infinite limit at $+\infty$, or to grow without bound, if any interval $]M;+\infty|$ contains all the values of $f(x)$ for x large enough - that is for x in an interval $]A;+\infty[$. It is written :

$$\lim_{x\to+\infty} f(x) = +\infty$$



Note : That means that the function f is not bounded above

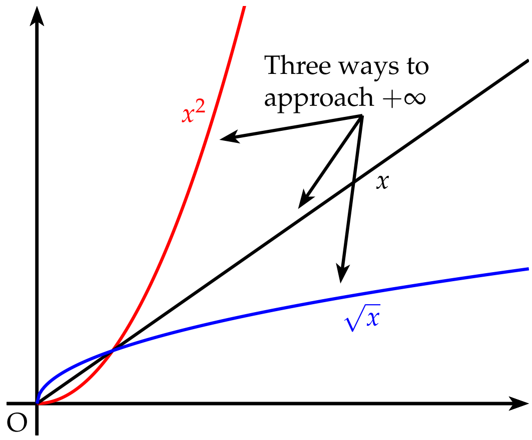
- An infinite limit at $-\infty$ is likewise defined and is denoted $\lim_{x\to-\infty} f(x) = +\infty$.
- Similarly : $\lim_{x\to+\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$

Example : The common functions : $x \mapsto x$, $x \mapsto x^n$ and $x \mapsto \sqrt{x}$ have a limit of $+\infty$ at $+\infty$.

The common function : $x \mapsto x^n$ has a limit of $+\infty$ at $-\infty$ if n is even and $-\infty$ at $-\infty$ if n is odd.

A function can approach $+\infty$ at $+\infty$ in several ways. For instance, consider the common functions $x \mapsto x^2$, $x \mapsto x$ and $x \mapsto \sqrt{x}$.

- $x \mapsto x^2$ approaches infinity "*rapidly*". The function is convex downward.
- $x \mapsto x$ approaches infinity "*moderately*" . There is no concavity.
- $x \mapsto \sqrt{x}$ approaches infinity "*slowly*". The function is convex upwards.

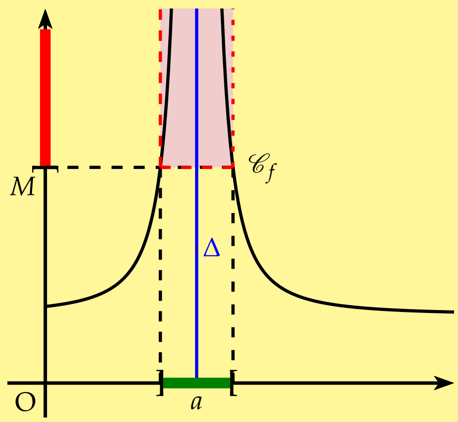


2 Infinite limit at a finite point

Definition 3 : A function f is said have a limit of $+\infty$ at a, if any interval $]M;+\infty[$ contains all the values of $f(x)$ for x is close enough to a - that is to say for x in an open interval containing a. It is denoted :

$$\lim_{x \to a} f(x) = +\infty$$

The line Δ defined by $x = a$ is the **vertical asymptote** of the graph $\mathscr{C}_f$

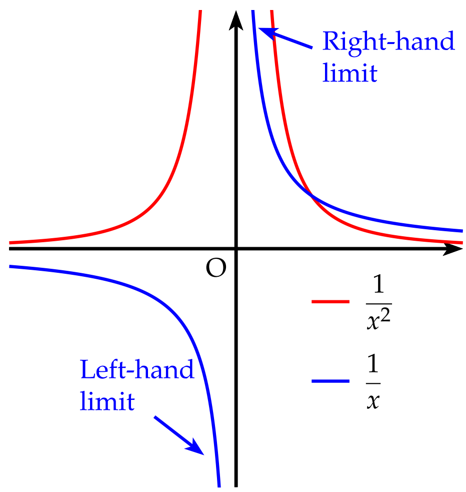


Note : A limit of $-\infty$ s likewise defined and denoted $\lim_{x \to a} f(x) = -\infty$

If the limit at $x = a$ does not exist, the function may still have a left-hand or right-hand limit at $x = a$. They are denoted as follows :

left-hand limit : $\lim_{\substack{x \to a \\ x < a}} f(x)$ or $\lim_{x \to a^-} f(x)$

right-hand limit : $\lim_{\substack{x \to a \\ x > a}} f(x)$ or $\lim_{x \to a^-} f(x)$



Examples : The function $x \mapsto \dfrac{1}{x^2}$ has a limit of $+\infty$ at 0.

The function $x \mapsto \dfrac{1}{x}$ has no limit at 0, but has a left- and right-hand limits at 0 – $-\infty$ and $+\infty$. The left- and right-hand limits are the same if and only if the ordinary limit exists

3 Limits of common functions

Limits at infinity

$f(x)$	x^n	$\dfrac{1}{x^n}$	$\sqrt{x}$	$\dfrac{1}{\sqrt{x}}$
$\lim\limits_{x\to+\infty} f(x)$	$+\infty$	0	$+\infty$	0
$\lim\limits_{x\to-\infty} f(x)$	$+\infty$ if n even $-\infty$ if n odd	0	not defined	not defined

Limits at 0

$f(x)$	$\dfrac{1}{x^n}$	$\dfrac{1}{\sqrt{x}}$
$\lim\limits_{\substack{x\to0\\x>0}} f(x)$	$+\infty$	$+\infty$
$\lim\limits_{\substack{x\to0\\x<0}} f(x)$	$+\infty$ if n even $-\infty$ if n odd	not defined

4 Operations on limits

4.1 Sum of functions

If f has a limit	ℓ	ℓ	ℓ	$+\infty$	$-\infty$	$+\infty$
If g has a limit	ℓ'	$+\infty$	$-\infty$	$+\infty$	$-\infty$	$-\infty$
then $f+g$ has a limit	$\ell+\ell'$	$+\infty$	$-\infty$	$+\infty$	$-\infty$	Ind.F.

Examples :

1) Limit at $+\infty$ of the function f defined on $\mathbb{R}^*$ by : $f(x) = x + 3 + \dfrac{1}{x}$

$$\left.\begin{array}{l}\lim\limits_{x\to+\infty} x+3 = +\infty \\ \lim\limits_{x\to+\infty} \dfrac{1}{x} = 0\end{array}\right\}$$

The sum is $\lim\limits_{x\to+\infty} f(x) = +\infty$

2) Limit at $+\infty$ and $-\infty$ of the function f defined on $\mathbb{R}$ by : $f(x) = x^2 + x$

$$\left.\begin{array}{l}\lim\limits_{x\to+\infty} x^2 = +\infty \\ \lim\limits_{x\to+\infty} x = +\infty\end{array}\right\}$$

The sum is $\lim\limits_{x\to+\infty} f(x) = +\infty$

$$\left.\begin{array}{l}\lim\limits_{x\to-\infty} x^2 = +\infty \\ \lim\limits_{x\to-\infty} x = -\infty\end{array}\right\}$$

The sum is not defined
Indeterminate form : $+\infty - \infty$

4.2 Product of functions

If f has a limit	ℓ	$\ell \neq 0$	0	∞
If g has a limit	ℓ'	∞	∞	∞
then $f \times g$ has a limit	$\ell \times \ell'$	∞*	Ind.F.	∞*

*Follow the usual rules on multiplying unlike or like signs

Examples :

1) The limit at $-\infty$ of the previous function : $f(x) = x^2 + x$

We change the expression of $f(x)$ to remove the indeterminate form, .

$$f(x) = x^2 + x = x^2\left(1 + \frac{1}{x}\right)$$

We then have :

$$\left.\begin{array}{l} \lim\limits_{x \to -\infty} x^2 = +\infty \\ \lim\limits_{x \to -\infty} 1 + \dfrac{1}{x} = 1 \end{array}\right\} \begin{array}{l} \text{The product is} \\ \lim\limits_{x \to -\infty} f(x) = +\infty \end{array}$$

2) The limit at $+\infty$ of the function defined on $\mathbb{R}_+$ by : $f(x) = x - \sqrt{x}$

We cannot summ the limits because it is an indeterminate form, we must change the expression of $f(x)$

$$f(x) = x - \sqrt{x} = x\left(1 - \frac{1}{\sqrt{x}}\right)$$

$$\left.\begin{array}{l} \lim\limits_{x \to +\infty} x = +\infty \\ \lim\limits_{x \to +\infty} 1 - \dfrac{1}{\sqrt{x}} = 1 \end{array}\right\} \begin{array}{l} \text{The product is} \\ \lim\limits_{x \to +\infty} f(x) = +\infty \end{array}$$

3) Right-hand limit at 0 of the function defined on $\mathbb{R}^*$ by : $f(x) = \dfrac{1}{x} \sin x$

$$\left.\begin{array}{l} \lim\limits_{\substack{x \to 0 \\ x > 0}} \dfrac{1}{x} = +\infty \\ \lim\limits_{\substack{x \to 0 \\ x > 0}} \sin x = 0 \end{array}\right\} \begin{array}{l} \text{Ind.F. } (0 \times \infty) \\ \textbf{use another method} \end{array}$$

4.3 Quotient of functions

If f has a limit	ℓ	$\ell \neq 0$	0	ℓ	∞	∞
If g has a limit	$\ell' \neq 0$	0 (1)	0	∞	ℓ' (1)	∞
then $\dfrac{f}{g}$ has a limit	$\dfrac{\ell}{\ell'}$	∞^*	Ind.F.	0	∞^*	Ind.F..

*Follow the usual rules on dividing like or unlike signs (1) without changing sign

Examples :

1) The limit at -2 of the function defined on $\mathbb{R} - \{-2\}$ by : $f(x) = \dfrac{2x - 1}{x + 2}$

We have the table of signs of $x + 2$:

x	$-\infty$		-2		$+\infty$
$x + 2$		$-$	0	$+$	

$$\left.\begin{array}{l}\lim\limits_{x\to -2} 2x-1=-5\\ \lim\limits_{\substack{x\to -2\\ x>-2}} x+2=0^+\\ \lim\limits_{\substack{x\to -2\\ x<-2}} x+2=0^-\end{array}\right\}\quad \begin{array}{l}\text{The quotient is}\\ \lim\limits_{\substack{x\to -2\\ x>-2}} f(x)=-\infty\\ \lim\limits_{\substack{x\to -2\\ x<-2}} f(x)=+\infty\end{array}$$

The function has a vertical asymptote of $x=-2$.

2) Limit at $+\infty$ of the function f defined by : $f(x)=\dfrac{2x+1}{3x+2}$

As both the numerator and the denominator approach $+\infty$ at $+\infty$, we are faced with an indeterminate form : $\dfrac{\infty}{\infty}$. We have to change the expression of $f(x)$.

$$f(x)=\frac{2x+1}{3x+2}=\frac{x\left(2+\dfrac{1}{x}\right)}{x\left(3+\dfrac{2}{x}\right)}=\frac{2+\dfrac{1}{x}}{3+\dfrac{2}{x}}$$

We then have :

$$\left.\begin{array}{l}\lim\limits_{x\to +\infty} 2+\dfrac{1}{x}=2\\ \lim\limits_{x\to +\infty} 3+\dfrac{2}{x}=3\end{array}\right\}\quad \begin{array}{l}\text{The quotient is}\\ \lim\limits_{x\to +\infty} f(x)=\dfrac{2}{3}\end{array}$$

4.4 Conclusion

There are four indeterminate forms (as with the limits of sequences) where the operations on the limits cannot conclude. In the case of an indeterminate form, we have to factor out the term of the highest degree (for polynomials and the rational functions), to simplify, to multiply by a conjugate (for radical functions), to use a comparison theorem, to change variables . . .

5 Limit of a composite function

Theorem 1 : Let two functions f, g be given. Let a, b and c be real numbers or $\pm\infty$.

$$\text{If}\quad \lim_{x\to a} f(x)=b \quad\text{and}\quad \lim_{x\to b} g(x)=c \quad\text{then}\quad \lim_{x\to a} g\left[f(x)\right]=c$$

Examples : Calculate the following limits :

1) $\lim\limits_{x\to +\infty} h(x)$ with $h(x)=\sqrt{2+\dfrac{1}{x^2}}$

2) $\lim\limits_{x\to +\infty} k(x)$ with $k(x)=\cos\left(\dfrac{1}{x^2+1}\right)$

1) Consider $f(x) = 2 + \dfrac{1}{x^2}$ and $g(x) = \sqrt{x}$, hence : $h(x) = g[f(x)]$.
We then compute the limits :

$$\left.\begin{array}{l} \lim\limits_{x \to +\infty} 2 + \dfrac{1}{x^2} = 2 \\ \lim\limits_{x \to 2} \sqrt{x} = \sqrt{2} \end{array}\right\} \quad \begin{array}{l}\text{The composite function is} \\ \lim\limits_{x \to +\infty} h(x) = \sqrt{2}\end{array}$$

Note : It is possible to use a change of variable. Let :

$$X = 2 + \frac{1}{x^2} \quad \text{so} \quad h(x) = \sqrt{X}$$

Then :

$$\left.\begin{array}{l} \lim\limits_{x \to +\infty} X = \lim\limits_{x \to +\infty} 2 + \dfrac{1}{x^2} = 2 \\ \lim\limits_{X \to 2} \sqrt{X} = \sqrt{2} \end{array}\right\} \quad \begin{array}{l}\text{The composite function is} \\ \lim\limits_{x \to +\infty} h(x) = \sqrt{2}\end{array}$$

2) Consider $f(x) = \dfrac{1}{x^2+1}$ and $g(x) = \cos x$. We then have : $k(x) = g[f(x)]$.

$$\left.\begin{array}{l} \lim\limits_{x \to +\infty} \dfrac{1}{x^2+1} = 0 \\ \lim\limits_{x \to 0} \cos x = 1 \end{array}\right\} \quad \begin{array}{l}\text{The composite function is} \\ \lim\limits_{x \to +\infty} k(x) = 1\end{array}$$

Theorem 2 : Using the limit of function for limit of sequence

Let (u_n) be a sequence defined by : $u_n = f(n)$. f is then the function associate to the sequence (u_n). Let a be a real number or $\pm\infty$

$$\text{If} \quad \lim_{x \to +\infty} f(x) = a \quad \text{then} \quad \lim_{n \to +\infty} u_n = a$$

Example : Given (u_n) defined for all $n \in \mathbb{N}^*$ by : $u_n = \sqrt{2 + \dfrac{1}{n^2}}$.

Let f be the function defined on $]0; +\infty[$ by : $f(x) = \sqrt{2 + \dfrac{1}{x^2}}$.

We saw above that : $\lim\limits_{x \to +\infty} f(x) = \sqrt{2}$

Hence the sequence (u_n) converges to $\sqrt{2}$

⚠ The converse of this theorem is false. A sequence can converge without the associate function having a limit. Indeed :

Let f be a function defined on $\mathbb{R}$ by : $\begin{cases} f(x) = 2 & \text{if } x \in \mathbb{N} \\ f(x) = 1 & \text{else} \end{cases}$

The limit of f in $+\infty$ clearly does not exist as the sequence (u_n) defined by $u_n = f(n) = 2$ converges to 2 !

6 Theorems of comparison

Theorem 3 : Let f, g, and h be three functions defined on the domain $I =]b; +\infty[$ and ℓ a real number.

1) **« Squeeze or sandwich » theorem**
   If for all $x \in I$, we have : $g(x) \leqslant f(x) \leqslant h(x)$ **and if** :

$$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} h(x) = \ell \quad \text{then} \quad \lim_{x \to +\infty} f(x) = \ell$$

2) **Comparison theorem**
   If for all $x \in I$ we have : $f(x) \geqslant g(x)$ **and if** :

$$\lim_{x \to +\infty} g(x) = +\infty \quad \text{then} \quad \lim_{x \to +\infty} f(x) = +\infty$$

Note : There is a similar theorem at $-\infty$ with $I =]-\infty; b[$ and at a finite point a with I an open interval containing a.

Proof :

1) Squeeze theorem : at $+\infty$

   If : $\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} h(x) = \ell$

   Then according the definition of limits, any open interval J containing ℓ, contains all the values of $g(x)$ and $h(x)$ for x enough large.

   As $g(x) \leqslant f(x) \leqslant h(x)$, the same can be said for $f(x)$.

   Conclusion : $\lim_{x \to +\infty} f(x) = \ell$

2) Comparison theorem : at $+\infty$

   According the definition of limits, any open interval $]M; +\infty[$, contains all the values of $g(x)$ if x is large enough.

   As $f(x) \geqslant g(x)$ the same can be said for $f(x)$.

   Conclusion : $\lim_{x \to +\infty} f(x) = +\infty$

Examples :

1) Calculate the limit of $f(x) = \dfrac{\sin x}{x}$ at $+\infty$

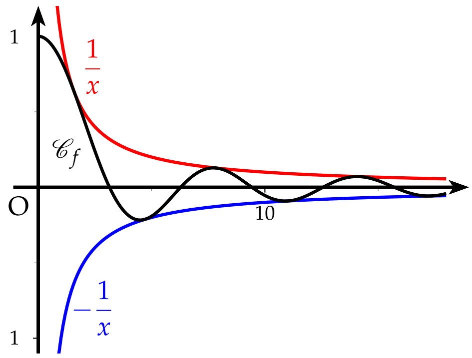
2) Calculate the limit of $g(x) = x + \cos x$ at $+\infty$

1) For all positive x , we have :
$-1 \leqslant \sin x \leqslant 1$, then :

$$\forall x > 0 \quad -\frac{1}{x} \leqslant f(x) \leqslant \frac{1}{x}$$

as :
$\lim\limits_{x \to +\infty} -\frac{1}{x} = 0$ and $\lim\limits_{x \to +\infty} \frac{1}{x} = 0$
so according to the squeeze theorem :

$$\lim_{x \to +\infty} f(x) = 0$$



2) We have : $\forall x \in \mathbb{R} \quad \cos x \geqslant -1$, then :

$$\forall x \in \mathbb{R} \quad x + \cos x \geqslant x - 1$$

As : $\lim\limits_{x \to +\infty} x - 1 = +\infty$, so according to the comparison theorem :

$$\lim_{x \to +\infty} g(x) = +\infty$$