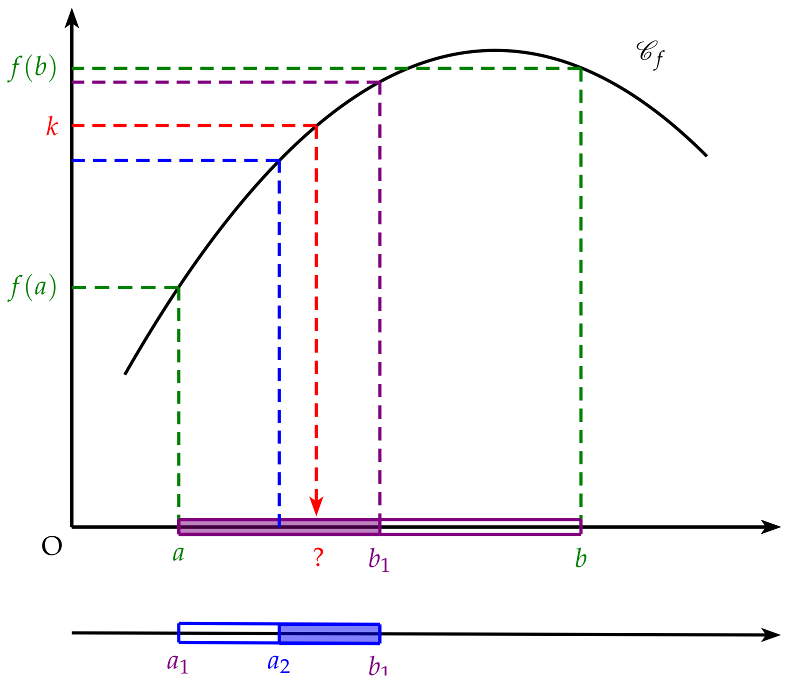
Proof of the Intermediate value theorem
The principal of dichotomy

1 The theorem

Theorem 1 : Let f be a **continuous** function defined on an interval I= $[a,b]$.
For any real number k between $f(a)$ and $f(b)$, there must be at least one value $c \in I$ such that $f(c) = k$.

2 The proof

Consider the following graph of f :



Construction of two adjacent sequences

Suppose $f(a) < f(b)$. (or else consider the function $g = -f$)
Construct two adjacent sequences (a_n) and (b_n) with following algorithm :

- If $m = \dfrac{a+b}{2}$ the middle of the interval $[a,b]$ is as $f(m) \geqslant k$ then :
  $a_1 = a$ and $b_1 = m$
- Else, $a_1 = m$ and $b_1 = b$.

We then have : $a \leqslant a_1 \leqslant b_1 \leqslant b$ and $f(a_1) \leqslant k \leqslant f(b_1)$

The process is then repeated :

- If $m = \dfrac{a_1 + b_1}{2}$ the middle of the interval $[a_1, b_1]$ satisfies $f(m) \geqslant k$ then :
  $a_2 = a_1$ and $b_2 = m$
- Else, $a_2 = m$ and $b_2 = b_1$.

We then have : $a \leqslant a_1 \leqslant a_2 \leqslant b_2 \leqslant b_1 \leqslant b$ and $f(a_2) \leqslant k \leqslant f(b_2)$

and so on, we therefore construct a sequence of segment growing smaller in size, and included one in the other :

$$[a, b] \supset [a_1, b_1] \supset \cdots \supset [a_n, b_n] \supset \ldots$$

The sequences (a_n) and (b_n) are respectively increasing and decreasing.

Furthermore, by construction, the length of the interval $[a_n, b_n]$ is $\dfrac{b-a}{2^n}$

So the segments $[a_n, b_n]$ have lengths which approach 0 and the sequences (a_n) and (b_n) are therefore adjacent sequences. So they converge to the same limit.

Let c be their common limit (the real number c is in the interval $[a, b]$)

Let us show that ***f(c) = k***

For any integer $n \in \mathbb{N}^*$: $f(a_n) \leqslant k \leqslant f(b_n)$

By taking the limit : $\lim\limits_{n \to +\infty} f(a_n) \leqslant k \leqslant \lim\limits_{n \to +\infty} f(b_n)$

As f is continuous at c : $f(c) \leqslant k \leqslant f(c)$

So : $f(c) = k$

Hence, we have shown that there is a real number c in $[a, b]$ such that $f(c) = k$.

Note : The hypothesis of continuity is essential in the theorem. Try to apply the intermediate value theorem to the « integer part » function with $a = 0$, $b = 1$ and $k = \dfrac{1}{2}$!!